International Research Journal of Innovations in Engineering and Technology (IRJIET)

ISSN (online): 2581-3048

Volume 4, Issue 7, pp 14-19, July-2020

https://doi.org/10.47001/IRJIET/2020.407003

The Multiplication Operation of Two Continued Fractions with Positive Non Integer Numerators

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Abstract **- This paper is a sequel to our previous work in which we found a elementary arithmetic operators of simple continued fractions. We consider the continued fractions (C.F.). We found the multiplication operation of two continued fractions with positive non integer numerators. We apply our finding to many examples of continued fractions.**

Keywords: Continued fractions. Multiplication operation of continued fractions.

I. INTRODUCTION

Since the beginning of the 20th century continued fractions have made their appearances in other field. An example of their versatility: solving Gear-Ratio problem and solving indeterminate equations. Continued fractions have also been utilized within computer algorithms for computing rational approximations to some real number [1,2,4,5,9]. Continued fractions have been extensively studied and there is a large body of research related to them [3,10]. Our main study is the operations of the continued fractions, previous studies have shown that the operations of the simple continued fractions follow similar patterns to those of continued fractions [6,7,8]. Therefore it is important to make this comparison. In this paper, we concentrate on the multiplication operation of two continued fractions with positive non integer numerators. We apply our result form or many examples.

II. PRELIMINARIES

Definition 2.1: A continued fraction is an expression of the form $a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{\ddots + \cfrac{b_n}{a_n}}}}$ where $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ can be real or complex, and their numbers can be either finite or infinite. In this paper, we consider the continued fractions of the form $a_0 + \cfrac{z}{a_1 + \cfrac{z}{a_2 + \cfrac{z}{\ddots + \cfrac{z}{a_n}}}}$, where a_0 is an integer, $a_1, a_2, a_3, \ldots, a_n$ are positive integer and z is a positive non integer. We denote it by $[a_0 ; a_1, a_2, \ldots]_z$.

Example 2.1

(a) $1 + \cfrac{\frac{5}{2}}{3 + \cfrac{\frac{5}{2}}{7}} = [1; 3,7]_{\frac{5}{2}}$. (b) $-3 + \cfrac{\frac{1}{3}}{1 + \cfrac{\frac{1}{3}}{2}} = [-3; 1,2]_{\frac{1}{3}}$. (c) $2 + \cfrac{\frac{2}{3}}{1 + \cfrac{\frac{2}{3}}{2 + \cfrac{\frac{2}{3}}{\cdots}}} = [2; 1,2, \ldots]_{\frac{2}{3}}$

$(a), (b)$ have a finite number of terms, and (c) has an infinite number of terms.

Theorem 2.1: A number is rational if and only if it can be expressed as a finite C.F. [6].

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Example 2.2:

(a) $\frac{40}{11} = 3 + \frac{\frac{3}{2}}{\frac{33}{14}} = 3 + \frac{\frac{3}{2}}{2+\frac{\frac{3}{2}}{\frac{21}{5}}} = 3 + \frac{\frac{3}{2}}{2+\frac{\frac{3}{2}}{4+\frac{\frac{3}{2}}{\frac{15}{2}}}} = 3 + \frac{\frac{3}{2}}{2+\frac{\frac{3}{2}}{4+\frac{\frac{3}{2}}{7+\frac{\frac{3}{2}}{3}}}} = [3,2,4,7,3]_{3/2}.$

(b) $\frac{28}{11} = 2 + \frac{\frac{2}{3}}{\frac{11}{9}} = 2 + \frac{\frac{2}{3}}{1+\frac{\frac{2}{3}}{3}} = [2\,,1,3]_{2/3}.$

(c) $\frac{3}{7} = 0 + \frac{\frac{3}{2}}{\frac{7}{2}} = 0 + \frac{\frac{3}{2}}{3+\frac{\frac{3}{2}}{3}} = [0,3,3]_{3/2}.$

(d) $\frac{7}{9} = 0 + \frac{\frac{7}{9}}{1} = [0;1]_{7/9}.$

Remark 2.1: To expand a negative rational number $-\frac{a}{b}\,(a\,,b > 0)$ into C.F we take the greatest integer number $\left\lfloor\frac{-a}{b}\right\rfloor$ for the first term of C.F that is, $\left\lfloor\frac{-a}{b}\right\rfloor = a_o'$, where $-a_0' \leq -\frac{a}{b} < -a_1' + 1$. We write $-\frac{a}{b} = -a_0' + \frac{z}{\frac{a'}{b'}}$ and then we use the same techniques in theorem 2.1 to get the remaining terms for $\frac{a'}{b'}$. That is, if $\frac{a'}{b'} = \left[a_1', a_2' \cdots, a_n'\right]_z$. Then $\frac{-a}{b} = \left[a_0'; a_1', a_2' \cdots, a_n'\right]_z$. [8].

Example 2.3: $-\frac{71}{11} = -7 + \frac{6}{11} = -7 + \frac{\frac{2}{3}}{\frac{11}{9}} = -7 + \frac{\frac{2}{3}}{1+\frac{\frac{2}{3}}{3}} = [-7,1,3]_{2/3}.$

Lemma 2.1: $[c_0;c_1,\ldots,c_{j-1},0,c_{j+1},\ldots,c_n]_z = [c_0;c_1,\ldots,c_{j-2},c_{j-1}+c_{j+1},c_{j+2},\ldots,c_n]_z$

III. MULTIPLICATION OPERATION FOR TWO CONTINUED FRACTIONS

Definition 3.2: Let $[a_0;a_1,\ldots,a_m]_z$ and $[b_0;b_1,\ldots,b_n]_z$ be two C.F. a_0, b_o are non-negative, then we defined their multiplication by:

(1) If $m = n$ then $[a_0;a_1,\ldots,a_n]_z \times \left[b_0;b_1,\ldots,b_n\right]_z = [d_0;d_1,\ldots,d_n]_z$ (3.2a)

Where

$d_0 = a_0 b_0.$

$d_1 = \left\lfloor\frac{a_1 b_1}{a_0 a_1 + b_0 b_1 + z}\right\rfloor.$

$d_2 = \left\lfloor\frac{z(a_0 b_2(a_1 a_2+z)+a_2 b_0(b_1 b_2+z)+z a_2 b_2)}{(a_1 a_2+z)(b_1 b_2+z)-d_1(a_o b_2(a_1 a_2+z)+a_2 b_0(b_1 b_2+z)+z a_2 b_2)}\right\rfloor.$

$\vdots \qquad \vdots \qquad \vdots$

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$$d_m = \left\lfloor \frac{z\left(k_m(a_1)k_m(b_1)k_{m-3}(d_2) - \left(a_0 k_m(a_1)k_{m-1}(b_2) + b_0 k_{m-1}(a_2)k_m(b_1) + z a_m b_m k_{m-2}(a_2)k_{m-2}(b_2)\right)k_{m-2}(d_1)\right)}{\left(a_0 k_m(a_1)k_{m-1}(b_2) + b_0 k_m(b_1)k_{m-1}(a_2) + z a_m b_m k_{m-2}(a_2)k_{m-2}(b_2)\right)k_{m-1}(d_1) - k_m(a_1)k_m(b_1)k_{m-2}(d_2)} \right\rfloor, \text{ if } m \text{ odd}$$

$$d_m = \left\lfloor \frac{z\left(\left(a_0 k_{m-1}(b_2)k_m(a_1) + b_0 k_{m-1}(a_2)k_m(b_1) + z a_m b_m k_{m-2}(a_2)k_{m-2}(b_2)\right)k_{m-2}(d_1) - k_m(a_1)k_m(b_1)k_{m-3}(d_2)\right)}{k_m(a_1)k_m(b_1)k_{m-2}(d_2) - \left(a_0 k_{m-1}(b_2)k_m(a_1) + b_0 k_{m-1}(a_2)k_m(b_1) + z a_m b_m k_{m-2}(a_2)k_{m-2}(b_2)\right)k_{m-2}(d_1)} \right\rfloor, \text{ if } m \text{ even,}$$

for $m = 2,3,\ldots,n$. The last term d_n is to expanded again as a C.F .

(2) If $m \neq n$ then (suppose that $m < n$) then

$[a_0;a_1,\ldots,a_m]_z \times [b_0;b_1,\ldots,b_m,b_{m+1},\ldots,b_n]_z = \left[d_0';d_1',\ldots,d_m',d_{m+1}',\ldots,d_n'\right]_z$ (3.2b) where $d_j' = d_j$ for $j = 1,2,\ldots,m$, (d_j as we did for case $m = n$), while $d_j' = d_{j,j-m}$

$$d_{j,j-m} = \left\lfloor \frac{z\left(k_m(a_1)k_j(b_1)k_{j-3}(d_2) - \left(a_0 k_m(a_1)k_{j-1}(b_2) + b_0 k_j(b_1)k_{m-1}(a_2) + z k_{m-1}(a_2)k_{j-1}(b_2)\right)k_{j-2}(d_1)\right)}{\left(a_0 k_m(a_1)k_{j-1}(b_2) + b_0 k_j(b_1)k_{m-1}(a_2) + z k_{m-1}(a_2)k_{j-1}(b_2)\right)k_{j-1}(d_1) - k_m(a_1)k_j(b_1)k_{j-2}(d_2)} \right\rfloor, \text{ if } j \text{ is odd}$$

$$d_{j,j-m} = \left\lfloor \frac{z\left(\left(a_0 k_m(a_1)k_{j-1}(b_2) + b_0 k_j(b_1)k_{m-1}(a_2) + z k_{m-1}(a_2)k_{j-1}(b_2)\right)k_{j-2}(d_1) - k_m(a_1)k_j(b_1)k_{j-3}(d_2)\right)}{k_m(a_1)k_j(b_1)k_{j-2}(d_2) - \left(a_o k_m(a_1)k_{j-1}(b_2) + b_0 k_j(b_1)k_{m-1}(a_2) + z k_{m-1}(a_2)k_{j-1}(b_2)\right)k_{j-1}(d_1)} \right\rfloor, \text{ if } j \text{ is even, for}$$

$j = m+1\,, m+2,\ldots,n$. Last term is to be treated as continued fractions as we did before, and not as greatest integer number.

Example 3.1: Find $[3;3]_{3/2} \times [3;7]_{3/2}$

Solution: Let $[3;3]_{3/2} = [a_0;a_1]_z$ and $[3;7]_{3/2} = [b_0;b_1]_z$, $(m = n = 1)$, from (3.2a), we have $[3;3]_{3/2} \times [3;7]_{3/2} = [a_0;a_1]_z \times [b_0;b_1]_z = [d_o;d_1]_z$, where d_1 is the last term and

$d_0 = a_0 b_0 = 3 \cdot 3 = 9.$

$d_1 = \frac{a_1 b_1}{a_0 a_1 + b_0 b_1 + z} = \frac{3 \cdot 7}{3 \cdot 3 + 3 \cdot 7 + \frac{3}{2}} = \frac{2}{3}$ (to be treated as C. F).

$= [0;2,6]_{3/2}.$

Then $[3;3]_{3/2} \times [3;7]_{3/2} = [9;0,2,6]_{3/2}$ (by lemma 2.1)

$= [11;6]_{3/2}.$

Example 3.2: Find $[3;3,3]_{3/2} \times [6;1,2]_{3/2}$.

Solution: Let $[3;3,3]_{3/2} = [a_0;a_1,a_2]_{3/2}$ and $[6;1,2]_{3/2} = [b_0;b_1,b_2]_{3/2}$, $m = n = 2$ from (3.2a) we have $[3;3,3]_{3/2} \times [6;1,2]_{3/2} = [d_0;d_1,d_2]_z$, where d_2 is the last term and

$d_0 = a_0 b_0 = 3 \cdot 6 = 18.$

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$$d_1 = \left\lfloor \frac{a_1 b_1}{a_0 a_1 + b_0 b_1 + z} \right\rfloor = \left\lfloor \frac{3}{9+6+\frac{3}{2}} \right\rfloor = \left\lfloor \frac{2}{11} \right\rfloor = 0.$$

$$d_2 = \frac{z(a_0 b_2 (a_1 a_2 + z) + a_2 b_0 (b_1 b_2 + z) + z a_2 b_2)}{(a_1 a_2 + z)(b_1 b_2 + z) - d_1 (a_o b_2 (a_1 a_2 + z) + a_2 b_0 (b_1 b_2 + z) + z a_2 b_2)},$$

$$= \frac{\frac{3}{2}\left(3\cdot 2\left(9+\frac{3}{2}\right)+6\cdot 3\left(2+\frac{3}{2}\right)+6\cdot\frac{3}{2}\right)}{\left(9+\frac{3}{2}\right)\left(2+\frac{3}{2}\right)-0\left(3\cdot 2\left(9+\frac{3}{2}\right)+6\cdot 3\left(2+\frac{3}{2}\right)+6\cdot\frac{3}{2}\right)} = \frac{270}{49} \qquad \text{(to be treated as C.F), we get}$$

$$\frac{270}{49} = 5 + \cfrac{\frac{3}{2}}{2+\cfrac{\frac{3}{2}}{1+\cfrac{\frac{3}{2}}{2+\cfrac{\frac{3}{2}}{2+\cfrac{\frac{3}{2}}{1+\cfrac{\frac{3}{2}}{2+\cfrac{\frac{3}{2}}{6+\cfrac{\frac{3}{2}}{2+\cfrac{\frac{3}{2}}{2+\frac{\frac{3}{2}}{6}}}}}}}}}}.$$

Then $[3;3,3]_{3/2} \times [6;1,2]_{3/2} = [18;0,5,2,1,2,2,1,2,6,2,2,6]_{3/2}$ (by lemma 2.1)

$= [23;2,1,2,2,1,2,6,2,2,6]_{3/2}$.

Example 3.3: Find $[3;3]_{3/2} \times [3;3,3]_{3/2}$

Solution: Let $[3;3]_{3/2} = [a_0;a_1]_z$ and $[3;3,3]_{3/2} = [b_0;b_1,b_2]_z$, $m = 1$, $n = 2$ from definition (3.2b) we get $[3;3]_{3/2} \times [3;3,3]_{3/2} = [a_0;a_1]_z \times [b_0;b_1,b_2]_z = \left[d_0';d_1',d_2'\right]_z$, where d_2' is the last term and

$$d_0' = d_0 = a_0 b_0 = 3 \cdot 3 = 9.$$

$$d_1' = d_1 = \left\lfloor \frac{a_1 b_1}{a_0 a_1 + b_0 b_1 + z} \right\rfloor = \left\lfloor \frac{9}{3\cdot 3+3\cdot 3+\frac{3}{2}} \right\rfloor = \left\lfloor \frac{18}{39} \right\rfloor = 0.$$

$$d_2' = d_2 = \left\lfloor \frac{z(a_0 a_1 b_2 + b_0(b_1 b_2 + z) + z b_2)}{a_1(b_1 b_2 + z) - d_1(a_0 a_1 b_2 + b_0(b_1 b_2 + z) + z b_2)} \right\rfloor = \left\lfloor \frac{\frac{3}{2}\left(27+3\left(9+\frac{3}{2}\right)+\frac{3}{2}\cdot 3\right)}{3\left(9+\frac{3}{2}\right)} \right\rfloor = \left\lfloor \frac{189}{63} \right\rfloor = 3.$$

Then $[3;3]_{3/2} \times [3;3,3]_{3/2} = [9;0,3]_{3/2}$ (by lemma 2.1)

$= [12]_{3/2} = 12.$

To check, we have $[3;3]_{3/2} \times [3;3,3]_{3/2} = \frac{7}{2} \cdot \frac{24}{7} = \frac{24}{2} = 12$

Example 3.4: Find $[4;2]_{7/4} \times [2;9,5,7]_{7/4}$.

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Solution: Let $[4;2]_{7/4} = [a_0;a_1]_z$, $[2;9,5,7]_{7/4} = [b_0;b_1,b_2,b_3]_z$, $m = 1$, $n = 3$ $(m < n)$ from definition (3.2b) we have $[4;2]_{7/4} \times [2;9,5,7]_{7/4} = [a_0;a_1]_z \times [b_0;b_1,b_2,b_3]_z = [d_0';d_1',d_2',d_3']_z$, where d_3' is the last term and

$$d_0' = d_0 = a_o b_0 = 4 \cdot 2 = 8.$$

$$d_1' = d_1 = \left\lfloor \frac{a_1 b_1}{a_0 a_1 + b_0 b_1 + z} \right\rfloor = \left\lfloor \frac{2\cdot 9}{4\cdot 2 + 2\cdot 9 + \frac{7}{4}} \right\rfloor = \left\lfloor \frac{72}{111} \right\rfloor = 0.$$

$$d_2' = \left\lfloor \frac{z(a_0 a_1 b_2 + b_0(b_1 b_2 + z) + z b_2)}{a_1(b_1 b_2 + z) - d_1(a_0 a_1 b_2 + b_0(b_1 b_2 + z) z b_2)} \right\rfloor = \left\lfloor \frac{\frac{7}{4}\left(4\cdot 2\cdot 5 + 2\left(45 + \frac{7}{4}\right) + \left(\frac{7}{4}\right)(5)\right)}{2\left(45 + \frac{7}{4}\right) - 0\left(4\cdot 2\cdot 5 + \left(45 + \frac{7}{4}\right) + \left(\frac{7}{4}\right)(5)\right)} \right\rfloor = \left\lfloor \frac{3983}{1496} \right\rfloor = 2.$$

$$d_3' = \frac{z\left(a_1(b_1 b_2 b_3 + z b_1 + z b_3)(1) - (a_0 a_1(b_2 b_3 + z) + b_0(b_1 b_2 b_3 + z b_1 + z b_3)(1) + (1)(b_2 b_3 + z) z) d_1'\right)}{\left(a_0 a_1(b_2 b_3 + z) + b_0(b_1 b_2 b_3 + z b_1 + z b_3)(1) + z(1)(b_2 b_3 + z)\right)\left(d_1' d_2' + z\right) - a_1(b_1 b_2 b_3 + z b_1 + z b_3)},$$

$$= \frac{\frac{7}{4}\left(2\left(315 + 9\left(\frac{7}{4}\right) + (7)\left(\frac{7}{4}\right)\right)\right)}{\left((4)(2)\left(35 + \frac{7}{4}\right) + 2\left(315 + (9)\left(\frac{7}{4}\right) + (7)\left(\frac{7}{4}\right)\right) + \left(35 + \frac{7}{4}\right)\left(\frac{7}{4}\right)\right)\left(\frac{7}{4}\right) - (2)\left(315 + (9)\left(\frac{7}{4}\right) + (7)\left(\frac{7}{4}\right)\right)(2)} = \frac{224}{85}.$$

To be treated as C. F.

$$\frac{224}{85} = [2; 2,2,5,3,2,1,1,2,22,2,3,1,2,6,4,3,10,5,2,4,5,2,3,8,4,9,5,7]_{7/4}.$$

Then $[4;2]_{7/4} \times [2;9,5,7]_{7/4}$

$= [8; 0,2,2,2,2,5,3,2,1,1,2,22,2,3,1,2,6,4,3,10,5,2,4,5,2,3,8,4,9,5,7]_{7/4}$ (by lemma 2.1)

$= [10; 2,2,2,5,3,2,1,1,2,22,2,3,1,2,6,4,3,10,5,2,4,5,2,3,8,4,9,5,7]_{7/4}$

IV. CONCLUSION

This paper is the Second part for the operations of the continued fractions with positive non Integer numerators. In the first part, we discovered the definitions of addition and subtractions operations of continued fractions. In this part, we defined the multiplication operation of two continued fractions.

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https://doi.org/10.47001/IRJIET/2020.407003

Citation of this Article:

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