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Efficient Class of Ratio–Cum–Product Estimator of Population Mean in Two Phase Sampling in Presence of Two Auxiliary Variables with Application to Agricultural Data Sets

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***Abstract* - In order to minimize errors in survey estimation situation, it is imperative that we adopt a sampling method that is capable of giving an improved results and which will be adjudged to be a better representation of the population. In this study, efficient class of ratio–cum-product estimators of population mean $\bar{Y}$ in Two Phase Sampling in presence of two auxiliary variables x and z was proposed. Members of the proposed class of estimator were obtained by varying the scalars associated with the proposed estimator, from where it was observed that the estimator produces the traditional sample mean ratio estimator$\bar{y}$, the dual to Singh and Tailor (2005) estimator due to Tailor et al (2012), Singh and Tailor (2011) generalized version of the dual to ratio-cum-product estimator of the population mean $\bar{Y}$. Various attribute of the proposed estimator such as biases, relative biases, Mean Square Errors (MSEs), and optimal MSEs were derived for cases I and II to the quadratic form of Tailor's series approximation. Theoretical proposition for evaluating efficiency was established and empirical study was conducted using Four (4) Agricultural data sets to ascertain the veracity of the theoretical proposition, from where it was found out from the results that the estimators $T_{DS1}T_{DS3}$, T_{DS4} , T_{DS5} and T_{DS12}, were more efficient in case I and estimators' $T^{*}_{DS1}T^{*}_{DS3}$, T^{*}_{DS4}, T^{*}_{DS4} T^{*}_{DS9}, T^{*}_{DS12}andT^{*}_{DS13} were more efficient in case II, having been ascertained to have produced smaller MSEs. The gain in efficiency increases as the sample size decreases and was more significant in case II. Therefore, sub-sampling the second sample independent of the first sample is advantageous and recommended for appreciable gain in efficiency and superiority over the first phase in double sampling scheme.**

Keywords: Efficiency, Estimator, Double Sampling and Population Mean.

1. Introduction

When a population cannot be studied by method of complete enumeration or census procedure due to financial inadequacy, limited time, manpower., scarcity of units etc., and we resort to sampling method where a sample is drawn from the population for the purpose of making generalization or inferences about the population based on sample observation, the method of sampling always give rise to sampling error since only a portion of the population is observed or studied. Another cause for concern is that neither complete enumeration nor sampling gives an exact result. In order to minimize errors in surveys situation, it became imperative that we adopt a sampling method that is capable of giving an improved results and which will be adjudged to be a better representation of the population.

Commonly, interest has been on Simple Random Sampling (SRS) as a cost effective scheme for sample selection in most researches in the field of Sciences, Social Sciences, and surveys of natural resources. But SRS method usually suffers weak or loss of efficiency in the presence of high variation of data sets. To minimize this bad samples, efforts were concentrated on employing Two Phase Sampling Scheme.

Many sampling techniques depend on the possession of advance information about an auxiliary variable X. Ratio estimation particularly requires the knowledge of such advance information. When such information on the supplementary variable is not feasible, it is comparatively cheap to draw a preliminary sample where information (the mean) on the auxiliary variable alone is obtained.

In some studies, where the objective is to estimate other variable (say Y), it becomes pertinent to commit some of the resources meant for entire the study to the preliminary sample. Such sampling procedure is called double or two phase

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sampling. Two phase sampling was first advocated by Neyman (1938) to address the problem of strata size in stratified sampling; while the estimation of population mean in two phase sampling for the classical ratio estimator of Cochran (1940) was first advocated by Sukhatme (1962). Other authors who proposed ratio estimator in double sampling scheme include Hydiroglou and Sarndal (1998), Singh and Vishwakarma (2007), Singh and Espejo (2007), Singh and Choudhury (2012), etc.

The continuous search for better estimators of population mean in double sampling made several author to propose various estimators which were found to be more efficient under some conditions. Such authors include; Solanki and Singh (2013) Singh and Choudhury (2012), Chamu and Singh (2014), Kumar and Vishwakarma (2014), Handique (2012), Kalita and Singi (2013), Sahoo and Singh (2014), Yadov and Kadilar (2013), Al-Saleh and Darabseh (2017), Singh and Yadav(2018), Al-Omari eta'l (2021), Abioye etal' (2020), Vishwarkarma and Zeehman (2021), etcetara. In continuation of the search for a better method of estimating the population means in Two Phase Sampling Scheme, this research put forward an efficient classes of ratio-cum-product estimators for two auxiliary variables, that evaluates properties such as bias and mean square error to a degree desired when compared to some other existing estimators in two phase sampling scheme and therefore shallserves as a better alternative whenever efficiency is considered.

1.1 Objectives of the study

The main objective of the study is to propose an efficient class of ratio –cum-product estimators of population mean in Two Phase Sampling, in presence of two auxiliary variables. The following specific objectives shall be realized:

- To derive the attributes of the proposed estimator.
- To obtain optimal condition for the proposed estimator
- To develop theoretical propositions for evaluating efficiency of the proposed estimator in relation to the tradition sample mean of ratio estimator.
- To conduct empirical study using Agricultural data sets to authenticate the theoretical propositions of the research.

1.2 Significance of the study

This research work shall serve as a springboard and a reference material for potential researchers who may wish to research further in this area. The theoretical proposition for the efficiency comparison of the estimators in this research shall guide users especially in the field of Statistics on the choice of an estimator with high degree of desirability since the difficulty in evaluating estimator and choosing between alternatives is due in part to a lack of comparative knowledge of the performance of the estimators. Advantageously, the research shall provide a better alternative whenever efficiency of estimators for estimating population mean is considered. The Research shall be beneficial to the Ministry of Agriculture, the Government and other agencies such as the National Population Commission, National Bureau of Statistics, and the Central Bank of Nigeria in the formulation, planning and implementation of her policies and programmes.

2. Literature Review

2.1 Ratio and Ratio-cum-product Estimation of Population Mean in Two Phase Sampling

It is pertinent to note that when information on the supplementary variable is not feasible to obtain, the ratio estimation under simple random sampling becomes impossible. In such a case double or two phase sampling becomes imperative. Literatures on sampling is quite vast and traceable to the early part of the 20th century with a sketch of history of survey sampling by Kovalevsky (1924), Bowley (1926) laid the foundational stone of modern sampling dealing with stratification, Neyman (1934, 1938), advocated two-phase sampling to address the problem of strata sizes in stratified sampling. For more studies on comparison of sampling techniques; Aneesh, S. Hariharan, V., Gallucci, F., & Graig, H., (2013) carried out an estimation of relative efficiency of adaptive cluster versus traditional Sampling designs with application to arrivals of sharks Several authors who provided solutions to practical problems through the application of two phase (or double) sampling procedures include Spur (1952), Freese (1962), Unikrishan and Kunte (1995), Hidiroglou and Sarndal (1998), Singh and Espejo (2007), etc.

In order to improve the precision of ratio estimation under this sampling technique several authors have done much work using the supplementary variable. Thus, Srivastava (1970) modified the classical ratio estimator under two- phase sampling to obtain a better estimator. Also motivated by Bahl and Tuleja (1991), Singh and Vishwarkama (2007) and Singh, Kumar, and Smarandache (2008) proposed ratio estimator in double sampling and showed the condition under which their estimators were more efficient than some existing estimators. Singh and Choudhury (2012) proposed a product – cum – dual to ratio estimators in double sampling and found out that the estimator had equal efficiency as the regression estimator under this sampling scheme. Also motivated by Singh and Tailor (2003), Malik and Tailor (2013) obtained improved estimator of population mean in double sampling and showed conditions where the obtain estimator would have efficiency

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greater than the classical ratio estimator in two phase sampling.

Similarly, Gargele and Choubey (2013) derived a generalized family of ratio estimator in double sampling but failed to carry out an empirical study to support the theoretical findings of the study. Based on the availability of information on the supplementary variate at the first phase of sampling, Samiuddin and Hanif (2007) advocated three different estimators in double sampling. Parrott, Lhotka and Feridshouse (2012), and Olusengun (2013) made use of ratio estimators under two phase sampling to improve woody biomass estimation and Height of Tecno Grandis (THT) respectively. Solanki and Singh (2013) suggested a family of ratio estimator double sampling using single information on auxiliary variable and it properties were studied. The advocated estimator was shown to be more efficient than the usual ratio and product estimators.

Again, motivated by Bahl and Tuteja (1991) Yedav and Kadilar (2013) suggested a generalized estimator of population mean, studied its properties and obtained the asymptotic optimum estimator (AOE). Using the method of Singh, Chandra and Singh (1993) they obtained an almost unbiased ratio cum-product exponential estimator. Srivastava and Tracy (1980) introduce transformation of auxiliary variable to change negative correlation situations into a positive one vice- versa, giving rise to duals in ratio estimator. Singh (1982) gave a generalized transformation capable of dealing with positive and negative correlation simultaneously and derived a wide class of unbiased product-type estimators. More so, the study of ratio and regression estimators based on super-population models has been undertaken by Rao (1987).

However in many surveys of practical importance or where sensitive issues are of interest, information is generally not obtained from all samples units even after callbacks. Thus the estimation of population mean in the presence of non-response using single supplementary variate, has been considered by several authors, some of whom are; Srinath (1971), Rao (1987), Khare and Srivastatva (1993, 1995, 1997), Rathour (2012) etc., while the case of some missing observations was discussed by Toutenberg and Srivastava (1998) , Singh and Joarder (1998), Singh, Joarder and Tracy (2000), Singh and Tracy (2001) and Singh and Tracy (2003).

For obvious reason of improved precision, several authors are endeared to the use of supplementary variates to estimate certain population characteristic of interest in double sampling procedures. However, some notable references using two or more auxiliary variables with double sampling are Olkins (1958), Raj (1965), Srivastava (1965, 1971), Kumar and Vishwakarma (2014), etc. Raj (1965) considered a weighted difference estimator (in SRSWOR), similar to that of Olkin (1958) estimator and extended the result to two phase sampling. Srivastava (1965) and Rao and Mudholkar (1967) generalized the Olkin's (1958) estimator using a combination of positive and negative correlated auxiliary variables. Also Singh (1982), suggested three different ratio-cum-product estimators using supplementary variate that are correlated with the variate of interest. Srivastava (1971) presented a generalized ratio-type estimator where the sample mean and other existing estimators were obtained as particular cases of this estimator. Goswami and Sukhatme (1965) proposed and studied the double sampling version of Olkin's estimators for three stage sampling. More so, using two supplementary variate, Mohanty (1967) proposed a regression-cum-ratio estimator in two phase sampling scheme.

Other authors with notable contributions in the estimation of certain population characteristics using single or multiple auxiliary variables in two phase sampling include Singh and Tracy (2001) and Al-jaraha and Ahmed (2002). In this research work, we suggest a family of estimators of population mean in two phase sampling techniques using a combination of suitably chosen scalars and known population mean on a single supplementary varaiate.

Singh and Yadev (2018) developed a family of ratio – cum-product estimators for finite population mean $\bar{Y}$ of the study variable using information on two auxiliary variables (x, z). It was shown that the usual unbiased estimator $\bar{Y}$ ratio estimator, product estimator, dual to ratio estimator and dual to product estimator due to Srivenkatramana(1980), Bandyopadhyaya(1980), Singh et al (2005, 2011) estimators, Tailor e tal (2012) estimator, Vishwakarma Zeehman(2021) estimators, Vishwakarma and Kumar (2015) estimator are members of the suggested family of estimators. In addition to these estimators, various unknown estimators were shown to be members of the suggested family of estimators. The bias and Mean Square errors of the suggested family of estimators were obtained under large sample approximation. Efficiency comparisons were made to demonstrate the performance of the suggested family of estimator over other existing estimators and an empirical was carried out to support the study.

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3. Methodology

This study applies the method of mathematical expectation and Taylor's series approximation to the second order to derive the theoretical results. Some existing estimators (with their properties) that are related to this study are presented. A generalized efficient family of ratio estimator of population mean in double sampling is suggested.

The properties of the suggested estimator as well as its optimality condition were obtained. This condition was then used to obtain an expression for the Asymptotic Optimal Estimator (AOE), its bias and MSE.

3.1 Sampling method

(a) Simple Random Sampling description

In case of two auxiliary variables X and Z, when sampling is done on Z.

Let $\{U_1,\ U_2\,,\dots,U_M\}$ be a finite population having M units, where U_i is a set of values$(X_i,\ Y_i\,,Z_i)$, $i=1,2,\dots,M$, Here Y is a study variable, X and Z are auxiliary variables which is correlated with Y. Let$(y_1,\ y_2\,,\dots,y_m)$, $(x_1,\ x_2\,,\dots,x_{m,})$and $(z_1,$ $z_2\,,\dots,z_{m,})$be m sample values, then under Simple Random Sampling without Replacement (SRSWOR), the means and variances of the study and supplementary variables are given as:

$$\left.\begin{aligned}\bar{X}^{SRS} &= \frac{1}{m}\left(\sum_{i=1}^{m} X_i\right)\\ \bar{Y}^{SRS} &= \frac{1}{m}\left(\sum_{i=1}^{m} Y_i\right)\\ \bar{Z}^{SRS} &= \frac{1}{m}\left(\sum_{i=1}^{m} Z_i\right)\end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned}Var(\bar{X}^{SRS}) &= \left(\frac{1-f}{m}\right)\sigma_x^2\\ Var(\bar{Y}^{SRS}) &= \left(\frac{1-f}{m}\right)\sigma_y^2\\ Var(\bar{Z}^{SRS}) &= \left(\frac{1-f}{m}\right)\sigma_z^2\end{aligned}\right\} \quad (2)$$

If the finite population correction $f\neq 0$

$$\left.\begin{aligned}\rho_{XY} &= \frac{\sigma_{XY}}{\sigma_X\sigma_Y}, \quad \sigma_{XY} = Cov(\bar{Y}^{SRS},\bar{X}^{SRS}) = \rho_{XY}\sigma_X\sigma_Y\\ \rho_{XZ} &= \frac{\sigma_{XZ}}{\sigma_X\sigma_Y}, \quad \sigma_{XY} = Cov(\bar{X}^{SRS},Z^{SRS}) = \rho_{XZ}\sigma_X\sigma_Z\\ \rho_{YZ} &= \frac{\sigma_{YZ}}{\sigma_Y\sigma_Z}, \quad \sigma_{YZ} = Cov(\bar{Y}^{SRS},\bar{Z}^{SRS}) = \rho_{YZ}\sigma_Y\sigma_Z\end{aligned}\right\}(3)$$

$\rho_{xy,}$ is the correlation coefficient between X and Y, σ_{xy} is the covariance between X and Y
$\rho_{xz,}$ is the correlation coefficient between X and Z, σ_{xz} is the covariance between X and Z
ρ_{yz}, is the correlation coefficient between X and Z, σ_{yz} is the covariance between Y and Z

3.2 Double sampling description

Let $\pi_i=\{\pi_{1,}\pi_{2,}\pi_{3,}\cdots\pi_M\}$ be a population containing the study and supplementary variate taking values on the π_i. Two approaches or cases of estimating the population mean are presented below:

Case I: "A large preliminary sample of size m_1 is selected by simple random sampling without replacement (SRSWOR) from the population of M units and information is obtained on the supplementary variable alone. A second sub-sample of size $m_{2,}(m_2<m_1)$ is selected by simple random sampling SRSWOR. Information on Y is obtained from the second phase sub-sample".

Case II: A second sample of size m_2 is obtained from the population independent of the first phase sample and information on both the supplementary and study character are obtained from this sample.

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3.3 Biases, Relative Biases and Mean Square Errors of T_{DSj}, $i = 1, 2, 3 \ldots$

If T_{DSi}, $i = 1, 2, 3 \ldots$is the proposed efficient class of estimators of the Population mean $\bar{Y}$under two phase sampling scheme for two auxiliary variables X and Z, then the biases, relative biases and Means Square Errors (MSEs) shall be obtained using:

(a) Biases

$$1.\quad B(T_{DSi}) = [E(T_{DSi}) - \bar{Y}], i = 1,2, \ldots \tag{4}$$

(b) Relative Biases

$$(1). RB(T_{DSi}) = \frac{[E(T_{DSi}) - \bar{Y}]}{\bar{Y}}, i = 1,2, \ldots \tag{5}$$

(c) MSEs

$$1.\quad MSE(T_{DSi}) = E[(T_{DSi}) - \bar{Y}]^2,\ j = 1,2, \ldots \tag{6}$$

3.4 Definition

3.4.1: Dominant Estimator:

If T_{DSi} and T_{DSj} are estimators for estimating the parameter $\bar{Y}$, then $T_{DSi}$is said to be dominant T_{DSj}, if
i. The MSE of T_{DSi} is smaller for at least some values of $\bar{Y}$
ii. The MSE of T_{DSi} does not exceed that of T_{DSj}or any value of $\bar{Y}$
iii. $E[(T_{DSi}) - \bar{Y}]^2 \leq E\left[\left(T_{DSj}\right) - \bar{Y}\right]^2$ and vice versa if T_{DSj}dominates T_{DSi} , $j = 1, 2, \ldots$,

3.4.2: More Efficient Estimator

If T_{DSi} and T_{DSj} are estimators for estimating the parameter $\bar{Y}$, then T_{DSi} is said to be more efficient than T_{DSj}, if T_{DSi} dominates T_{DSj}

3.4.3: Most Efficient Estimator

If $T_{DS1}, T_{DS2}, T_{DS3}, T_{DS4}$, T_{DS4}, … are a collection of estimators for estimating the population parameter $\bar{Y}$, then the most efficient estimator is the one whose predictions have the least variance or Mean Square Error (MSE) in relation to its brethren or competitor's estimators.

3.4.4: Percentage Relative Efficiency

An estimator is said to be percentage efficient in relation to its competitor's estimator if the estimator in question is the one with the smallest Percentage Relative Efficiency (PRE).

3.5 Theoretical underpinnings of comparison of efficiency

Let $MSE(T_{DSi})_{opt}$,$MSE\left(T_{DSj}\right)_{opt}$, be the MSEs of the proposed estimator in two phase sampling scheme and that of any of its members the following conditions holds.

3.5.1 Efficiency of $MSE(T_{DSi})_{opt}$ over $MSE\left(T_{DSj}\right)_{opt}$

(i) $MSE(T_{DSi})_{opt}$is more efficient than $MSE\left(T_{DSj}\right)_{opt}$ if,

$$\frac{MSE(T_{DSi})_{opt}}{MSE\left(T_{DSj}\right)_{opt}} < 1 \quad or \quad \frac{1}{\frac{MSE(T_{DSi})_{opt}}{MSE\left(T_{DSj}\right)_{opt}}} > 1 \text{, for } j = 1\ to\ m \tag{7}$$

3.5.2 Most Efficient Estimators

$MSE(T_{DSi})_{opt}$,is most efficient than $MSE(T_{DS2})_{opt}$and $MSE(T_{DS3})_{opt}$, …, if,

$$MSE(T_{DSi})_{opt} < MSE(T_{DS2})_{opt} < MSE(T_{DS3})_{opt} < \cdots \tag{8}$$

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3.5.3 Percent Relative Efficiency PRE

(i) $MSE(T_{DSi})_{opt}$ is more efficient than $MSE(T_{DSi})_{opt}$ in terms of PRE, if

$$\frac{MSE(T_{DSi})_{opt}}{MSE(T_{DSj})_{opt}} \times 100 < 100 \quad or \quad \frac{1}{\frac{MSE(T_{DSi})_{opt}}{MSE(T_{DSj})_{opt}}} \times 100 > 100 \text{ , for } j = 1\,,2,\ldots \qquad (9)$$

3.6 Proposed efficient classes of ratio –cum-product estimators of population mean in Two Phase Sampling in presence of two auxiliary variables

An Efficient classes of ratio –cum-product estimators of population mean in Two Phase Sampling in presence of two auxiliary variables and presented as follows:

$$T_{DS} = \bar{y}\left[w_1\left(\frac{a\bar{X}+\rho_{xz}}{a\bar{x}+\rho_{xz}}\right)^{\alpha_1}\left(\frac{bz+\rho_{xz}}{b\bar{Z}+\rho_{xz}}\right)^{\alpha_2} + w_2\left(\frac{a\bar{x}^*+\rho_{xz}}{a\bar{X}+\rho_{xz}}\right)^{\delta_1}\left(\frac{b\bar{Z}+\rho_{xz}}{b\bar{z}^*+\rho_{xz}}\right)^{\delta_2}\right] \qquad (10)$$

Where

$\bar{y} = \sum_{i=1}^{n}\frac{y_i}{m_1}$, the sample mean of the variable of interest obtained from the second phase sample

$\bar{x} = \sum_{i=1}^{m_1}\frac{x_i}{m_1}$, the first phase sample mean of the auxiliary variable

$\bar{z} = \sum_{i=1}^{m_2}\frac{z_i}{m_2}$, the second phase sample mean of the auxiliary variable

$\bar{Y} = \sum_{i=1}^{M}\frac{Y_i}{M}$, the unknown population mean of the study variable Y

$\bar{X} = \sum_{i=1}^{M}\frac{X_i}{M}$, the unknown population mean of the first auxiliary variable X

$\bar{Z} = \sum_{i=1}^{M}\frac{Z_i}{M}$, the unknown population mean of the second auxiliary variable Z

$(a, b \neq 0, \rho_{xz} \neq 0)$ are real numbers and also may take the values of parameters associated with either study variable y or the auxiliary variables(x, z); $(\alpha_1,\ \alpha_2,\ \delta_1, \delta_2)$ are scalars or real constants which helps in designing the estimators and can be determined suitably.$(w_1, w_2)$are suitably chosen scalars whose sum need not be unity. When α_1, α_2, δ_1, and δ_2 are fixed, w_1, w_2may be selected in an optimum manner by minimizing the (MSEs) of $T_{DS_i} i = 1\,,2,3,\ldots$ with respect to w_1, w_2. Where$\overline{x^*} = \{(1+g_1)\bar{X} - g_1\bar{x}\}$,

$\bar{z^*} = \{(1+g_2)\bar{Z} - g_2\bar{z}\}$are unbiased estimator of population means,$\bar{Z}^{ERSS}$,$\bar{Z}^{ERSS}$ respectively,

$$g_1 = \frac{m_1}{(M-m_1)} = \frac{f_1}{(1-f_1)} \qquad (11)$$

$$g_2 = \frac{m_2}{(M-m_2)} = \frac{f_2}{(1-f_2)} \qquad (12)$$

where $f_1 = \frac{m_1}{M}$, $f_2 = \frac{m_2}{M}$

3.7 Biases, MSEs and optimal MSEs of the proposed estimators

To obtain the bias and Mean Square Error of the class of estimators T_{Ds}we write

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$$\left.\begin{aligned}
\bar{y} &= \bar{Y}(1+e_y)\\
\bar{x} &= \bar{X}(1+e_x)\\
\bar{z} &= \bar{Z}(1+e_z)\\
E(e_y) &= E(e_x) = E(e_z) = 0\\
E(e_y^2) &= \theta\frac{Var(\bar{y})}{(\mu_y)^2} = C_y^2\\
E(e_x^2) &= \theta\frac{Var(\bar{x})}{(\mu_x)^2} = C_x^2\\
E(e_z^2) &= \theta'\frac{Var(\bar{z})}{(\mu_z)^2} = C_z^2\\
\overline{x^*} &= \{(1+g_1)\bar{X} - g_1\bar{x}\}\\
\overline{z^*} &= \{(1+g_2)\bar{Z} - g_2\bar{z}\}\\
E(e_ye_x) = C_{yx} &= \theta\left(\rho_{xy}\cdot\frac{\sqrt{Var(\bar{y})}}{\mu_y}\cdot\frac{\sqrt{var(\bar{x})}}{\mu_x}\right) = \rho_{yx}C_yC_x\\
E(e_ye_z) = C_{yz} &= \theta'\left(\rho_{yz}\cdot\frac{\sqrt{Var(\bar{y})}}{\mu_y}\cdot\frac{\sqrt{Var(\bar{z})}}{\mu_z}\right) = \rho_{yz}C_yC_z\\
E(e_xe_z) = C_{xz} &= \theta'\left(\rho_{xz}\cdot\frac{\sqrt{var(\bar{x})}}{\mu_x}\cdot\frac{\sqrt{Var(\bar{z})}}{\mu_z}\right) = \rho_{xz}C_xC_z\\
\theta &= \left(\frac{1-f_1}{m_1}\right)\\
\theta' &= \left(\frac{1-f_2}{m_2}\right)
\end{aligned}\right\} \quad (13)$$

Transforming the equation (10) into an expandable form we have

$$T_{DS} = \bar{Y}(1+e_y)\left[w_1\left(\frac{a\bar{X}+\rho_{xz}}{a\bar{X}(1+e_x)+\rho_{xz}}\right)^{\alpha_1}\left(\frac{b\bar{Z}(1+e_z)+\rho_{xz}}{b\bar{Z}+\rho_{xz}}\right)^{\alpha_2} + w_2\left(\frac{a\{(1+g_1)\bar{X}-g_1\bar{x}\}+\rho_{xz}}{a\bar{X}+\rho_{xz}}\right)^{\delta_1}\left(\frac{b\bar{Z}+\rho_{xz}}{b\{(1+g_2)\bar{Z}-g_2\bar{z}\}+\rho_{xz}}\right)^{\delta_2}\right]$$

$$T_{DS} = \bar{Y}(1+e_y)\left[w_1\left(1+\frac{a\bar{X}e_x}{a\bar{X}+\rho_{xz}}\right)^{-\alpha_1}\left(1+\frac{b\bar{Z}e_z}{b\bar{Z}+\rho_{xz}}\right)^{\alpha_2} + w_2\left(1-\frac{g_1a\bar{X}e_x}{a\bar{X}+\rho_{xz}}\right)^{\delta_1}\left(1-\frac{g_2b\bar{Z}e_z}{b\bar{Z}+\rho_{xz}}\right)^{-\delta_2}\right]$$

$$T_{DS} = \bar{Y}(1+e_y)\left[w_1\left(1+\frac{a\bar{X}e_x}{a\bar{X}+\rho_{xz}}\right)^{-\alpha_1}\left(1+\frac{b\bar{Z}e_z}{b\bar{Z}+\rho_{xz}}\right)^{\alpha_2} + w_2\left(1-\frac{g_1a\bar{X}e_x}{a\bar{X}+\rho_{xz}}\right)^{\delta_1}\left(1-\frac{g_2b\bar{Z}e_z}{b\bar{Z}+\rho_{xz}}\right)^{-\delta_2}\right]$$

$$T_{DS} = \bar{Y}(1+e_y)[w_1(1+\lambda_a e_x)^{-\alpha_1}(1+\lambda_b e_z)^{\alpha_2} + w_2(1-g_1\lambda_a e_x)^{\delta_1}(1-g_2\lambda_b e_z)^{-\delta_2}] \quad (14)$$

$$\lambda_a = \frac{a\bar{X}}{a\bar{X}+\rho_{xz}} \quad ,\lambda_b = \frac{b\bar{Z}}{b\bar{Z}+\rho_{xz}}$$

We authenticated the second order of approximation on the basis that the sample size is large enough to get

$|\lambda_a e_x| < 1$, $|\lambda_b e_z| < 1$, $|g_1\lambda_a e_x| < 1$ and $|g_2\lambda_b e_z| < 1$, so that $(1+\lambda_a e_x)^{-\alpha_1}$, $(1+\lambda_b e_z)^{-\alpha_2}$ $(1-g_1\lambda_a e_x)^{\delta_1}and(1-g_2\lambda_b e_z)^{-\delta_2}$ can be expanded up to the quadratic form of the Taylor's series approximation. Equation (14) can now be written as;

$$T_{DS} = \bar{Y}(1+e_y)\left[w_1\left((1-\alpha_1\lambda_a e_x + \frac{\alpha_1(\alpha_1+1)}{2}\lambda_a^2e_x^2 - \cdots\right)\left((1+\alpha_2\lambda_b e_z + \frac{\alpha_2(\alpha_2-1)}{2}\lambda_b^2e_z^2 + \cdots\right)\right] + \left[w_2\left((1+g_1\delta_1\lambda_a e_x + \frac{\delta_1(\delta_1-1)}{2}{g_1}^2\lambda_a^2e_x^2 + \cdots\right)\left((1-g_2\delta_2\lambda_b e_z - \frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_b^2e_z^2 + \cdots\right)\right]$$

$$T_{DSj} = \bar{Y}(1+e_y)\left[w_1(1+\alpha_2\lambda_b e_z + \frac{\alpha_2(\alpha_2-1)}{2}\lambda_b^2e_z^2 - \alpha_1\lambda_a e_x - \alpha_1\alpha_2\lambda_a\lambda_b e_xe_z + \frac{\alpha_1(\alpha_1+1)}{2}\lambda_a^2e_x^2 + \cdots\right] + \left[w_2\left((1-g_2\delta_2\lambda_b e_z - \frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_b^2e_z^2 + g_1\delta_1\lambda_a e_x - g_1g_2\delta_1\delta_2\lambda_a\lambda_b e_xe_z + \frac{\delta_1(\delta_1-1)}{2}{g_2}^2\lambda_a^2e_x^2 + \cdots\right)\right] \quad (15)$$

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By expanding the right-hand side of (15) and ignoring terms of $e's$ with exponents greater than two, gives:

$$T_{DS}=\bar{Y}\left[w_1(1+e_y+\alpha_2\lambda_b e_z+\alpha_2\lambda_b e_y e_z+\frac{\alpha_2(\alpha_2-1)}{2}\lambda_b^2 e_z^2-\alpha_1\lambda_a e_x-\alpha_1\lambda_a e_y e_x-\alpha_1\alpha_2\lambda_a\lambda_b e_x e_z+\frac{\alpha_1(\alpha_1+1)}{2}\lambda_a^2 e_x^2+\cdots\right]+\left[w_2\left((1+e_y-g_2\delta_2\lambda_b e_z-g_2\delta_2\lambda_b e_y e_z-\frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_b^2 e_z^2+g_1\delta_1\lambda_a e_x+g_1\delta_1\lambda_a e_y e_x-g_1g_2\delta_1\delta_2\lambda_a\lambda_b e_x e_z+\frac{\delta_1(\delta_1-1)}{2}{g_2}^2\lambda_a^2 e_x^2+\cdots\right)\right] \quad (16)$$

$$\left(\boldsymbol{T_{DSj}}-\bar{\boldsymbol{Y}}\right)=$$

$$\bar{Y}\,[w_1(1+e_y+\alpha_2\lambda_b e_z+\alpha_2\lambda_b e_y e_z+\frac{\alpha_2(\alpha_2-1)}{2}\lambda_b^2 e_z^2-\alpha_1\lambda_a e_x-\alpha_1\lambda_a e_y e_x-\alpha_1\alpha_2\lambda_a\lambda_b e_x e_z+\frac{\alpha_1(\alpha_1+1)}{2}\lambda_a^2 e_x^2+\cdots+w_2\left(1+e_y-g_2\delta_2\lambda_b e_z-g_2\delta_2\lambda_b e_y e_z-\frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_b^2 e_z^2+g_1\delta_1\lambda_a e_x+g_1\delta_1\lambda_a e_y e_x-g_1g_2\delta_1\delta_2\lambda_a\lambda_b e_x e_z+\frac{\delta_1(\delta_1-1)}{2}{g_2}^2\lambda_a^2 e_x^2+\cdots\right)-1\,] \quad (17)$$

To obtain the bias of the bias of the efficient class of ratio –cum-product estimators, we take the mathematical expectations of both sides (17) to give the Bias of the estimator $\boldsymbol{T_{DSj}}$ to the second order of approximation as:

$$\boldsymbol{E}\left(\boldsymbol{T_{DSj}}-\bar{\boldsymbol{Y}}\right)=\theta\bar{Y}\left[w_1\left(\frac{1}{\theta}+\alpha_2\lambda_b C_{yz}+\frac{\alpha_2(\alpha_2-1)}{2}\lambda_b^2 C_z^2-\alpha_1\alpha_2\lambda_a\lambda_b C_{xz}-\alpha_1\lambda_a\lambda_b C_{yx}+\frac{\alpha_1(\alpha_1+1)}{2}\lambda_a^2 C_x^2\right)+w_2\left(\frac{1}{\theta}-g_2\delta_2\lambda_b C_{yz}-\frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_b^2 C_z^2+g_1\delta_1\lambda_a C_{yx}-g_1g_2\delta_1\delta_2\lambda_a\lambda_b C_{xz}+\frac{\delta_1(\delta_1-1)}{2}{g_1}^2\lambda_a^2 C_x^2\right)-\frac{1}{\theta}\right] \quad (18)$$

To obtain the relative bias of the estimator, we use the expression;

$$\frac{\boldsymbol{E}(\boldsymbol{T_{DS}}-\bar{\boldsymbol{Y}})}{\bar{\boldsymbol{Y}}}=\theta\left[w_1\left(\frac{1}{\theta}+\alpha_2\lambda_b C_{yz}+\frac{\alpha_2(\alpha_2-1)}{2}\lambda_b^2 C_z^2-\alpha_1\alpha_2\lambda_a\lambda_b C_{xz}-\alpha_1\lambda_a\lambda_b C_{yx}+\frac{\alpha_1(\alpha_1+1)}{2}\lambda_a^2 C_x^2\right)+w_2\left(\frac{1}{\theta}-g_2\delta_2\lambda_b C_{yz}-\frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_b^2 C_z^2+g_1\delta_1\lambda_a C_{yx}-g_1g_2\delta_1\delta_2\lambda_a\lambda_b C_{xz}+\frac{\delta_1(\delta_1-1)}{2}{g_1}^2\lambda_a^2 C_x^2\right)-\frac{1}{\theta}\right] \quad (19)$$

By squaring both sides of equation (17) gives;

$$(\boldsymbol{T_{DS}}-\bar{\boldsymbol{Y}})^2=\bar{Y}^2 w_1^2(1+{e_y}^2+\alpha_1(\alpha_1+1)\lambda_a^2 e_x^2+{\alpha_1}^2\lambda_a^2 e_x^2+\alpha_2(\alpha_2-1)\lambda_b^2 e_z^2+{\alpha_2}^2\lambda_b^2 e_z^2-4\alpha_1\lambda_a e_y e_x+4\alpha_2\lambda_b e_y e_z-4\alpha_1\alpha_2\lambda_a\lambda_b e_x e_z)+\bar{Y}^2 w_2^2(\boldsymbol{1}+{e_y}^2++\delta_1(\delta_1-1){g_1}^2\lambda_a^2 e_x^2+{g_1}^2{\delta_1}^2\lambda_a^2 C_x^2-\delta_2(\delta_2+1){g_2}^2\lambda_b^2 e_z^2-{g_2}^2{\delta_2}^2\lambda_b^2 e_z^2-4g_2\delta_2\lambda_b e_y e_z+4g_1\delta_1\lambda_a e_y e_x-4g_1g_2\delta_1\delta_2\lambda_a\lambda_b e_x e_z)+\boldsymbol{2}\bar{Y}^2 W_1W_2\left(\boldsymbol{1}+{e_y}^2+\frac{\delta_1(\delta_1-1)}{2}{g_1}^2\lambda_a^2 e_x^2-\frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_b^2 e_z^2+(\frac{\alpha_2(\alpha_2-1)}{2}+\alpha_2 g_2\delta_2)\lambda_b^2 e_z^2+(\frac{\alpha_1(\alpha_1+1)}{2}-\alpha_1 g_1\delta_1)\lambda_a^2 e_x^2+(\alpha_2 g_1\delta_1-\alpha_1 g_2\delta_2)\lambda_a\lambda_b e_y e_z-2\alpha_1\lambda_a e_y e_x+2\alpha_2\lambda_b e_y e_z+2g_1\delta_1\lambda_a e_y e_x-g_1g_2\delta_1\delta_2\lambda_a\lambda_b e_x e_z-\alpha_1\alpha_2\lambda_a\lambda_b e_x e_z\right)-\boldsymbol{2}\bar{Y}^2 W_1\left(1+\frac{\alpha_1(\alpha_1+1)}{2}\lambda_a^2 e_x^2+\frac{\alpha_2(\alpha_2-1)}{2}\lambda_b^2 e_z^2-\alpha_1\lambda_a e_y e_x+\alpha_2\lambda_b e_y e_z-\alpha_1\alpha_2\lambda_a\lambda_b e_x e_z\right)-\boldsymbol{2}\bar{Y}^2 W_2\left(1-g_2\delta_2\lambda_b e_y e_z-\frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_b^2 e_z^2++g_1\delta_1\lambda_a e_y e_x-g_1g_2\delta_1\delta_2\lambda_a\lambda_b e_x e_z+\frac{\delta_1(\delta_1-1)}{2}{g_2}^2\lambda_a^2 e_x^2+\right)+\bar{Y}^2 \quad (20)$$

By taking the mathematical expectation of both sides of equation (20) gives the $\boldsymbol{MSE}\left(\boldsymbol{T_{DSj}}\right)$ **as;**

$$MSE(T_{DS})=E(T_{DS}-\bar{Y})^2=\bar{Y}^2[1+w_1^2(Q_1)+w_2^2(Q_2)+2W_1W_2(Q_3)-2W_1(Q_4)-2W_2(Q_5)] \quad (21)$$

Where,

$$\boldsymbol{Q_1}=\left(1+\theta\left[{c_y}^2+\alpha_1(2\alpha_1+1)\lambda_a^2 c_x^2+\alpha_2(2\alpha_2-1)\lambda_b^2 c_z^2-4\alpha_1\lambda_a C_{yx}+4\alpha_2\lambda_b C_{yz}-4\alpha_1\alpha_2\lambda_a\lambda_b C_{xz}\right]\right) \quad (22)$$

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$$\boldsymbol{Q_2} = \left(\mathbf{1} + \theta\left[c_y{}^2 + \delta_1(2\delta_1 - 1)g_1{}^2\lambda_a^2 c_x^2 + \delta_2(2\delta_2 + 1)g_2{}^2\lambda_b^2 c_z^2 + 4g_1\delta_1\lambda_a C_{yx} - 4g_2\delta_2\lambda_b C_{yz} - 4g_1 g_2\delta_1\delta_2\lambda_a\lambda_b C_{xz}\right]\right) \quad (23)$$

$$\boldsymbol{Q_3} = \left(\mathbf{1} + \theta\left[c_y{}^2 + \frac{\delta_1(\delta_1-1)}{2}g_1{}^2\lambda_a^2 c_x^2 - \frac{\delta_2(\delta_2+1)}{2}g_2{}^2\lambda_b^2 c_z^2 + \left(\frac{\alpha_2(\alpha_2-1)}{2} + \alpha_2 g_2\delta_2\right)\lambda_b^2 c_z^2 + \left(\frac{\alpha_1(\alpha_1+1)}{2} - \alpha_1 g_1\delta_1\right)\lambda_a^2 c_x^2 + (\alpha_2 g_1\delta_1 - \alpha_1 g_2\delta_2)\lambda_a\lambda_b C_{yz} - 2\alpha_1\lambda_a C_{yx} + 2\alpha_2\lambda_b C_{yz} + 2g_1\delta_1\lambda_a C_{yx} - \left(g_1 g_2\delta_1\delta_2 + \alpha_1\alpha_2\right)\lambda_a\lambda_b C_{xz}\right]\right) \quad (24)$$

$$\boldsymbol{Q_4} = \left(1 + \theta\left[\frac{\alpha_1(\alpha_1+1)}{2}\lambda_a^2 c_x^2 + \frac{\alpha_2(\alpha_2-1)}{2}\lambda_b^2 c_z^2 - \alpha_1\lambda_a C_{yx} + \alpha_2\lambda_b C_{yz} - \alpha_1\alpha_2\lambda_a\lambda_b\, C_{xz}\right]\right) \quad (25)$$

$$\boldsymbol{Q_5} = \left(1 + \theta\left[\frac{\delta_1(\delta_1-1)}{2}g_1{}^2\lambda_a^2 c_x^2 - \frac{\delta_2(\delta_2+1)}{2}g_2{}^2\lambda_b^2 c_z^2 + g_1\delta_1\lambda_a C_{yx} - g_2\delta_2\lambda_b C_{yz} - g_1 g_2\delta_1\delta_2\lambda_a\lambda_b C_{xz}\right]\right) \quad (26)$$

To obtain the optimum MSE of $\boldsymbol{T_{DS}}$ **,** we differentiate equation (21) with respect W_1, $W_2$and equate the result to zero. Thus;

$$\frac{\partial[MSE(T_{DSj})]}{\partial w_1} = 2w_1 Q_1 + 2w_2 Q_3 - 2Q_4 = 0 \quad (27)$$

$$\frac{\partial[MSE(T_{DSj})]}{\partial w_2} = 2w_2 Q_2 + 2w_1 Q_3 - 2Q_5 = 0 \quad (28)$$

Solving (27) and 28) simultaneously we have:

$$w_1 Q_1 + w_1 Q_3 + w_2 Q_2 + w_2 Q_3 - (Q_4 + Q_5) = 0$$

$$w_1(Q_1 + Q_3) + w_2(Q_2 + Q_3) = (Q_4 + Q_5) \quad (29)$$

From (27)

$$w_1 = \left(\frac{Q_4 - w_2 Q_3}{Q_1}\right) \quad (30)$$

Substituting (30) into (29), we have

$$\left(\frac{Q_4 - w_2 Q_3}{Q_1}\right)(Q_1 + Q_3) + w_2(Q_2 + Q_3) = (Q_4 + Q_5)$$

By making w_2 subject of the expression, we have

$$w_2 = \left(\frac{Q_1 Q_5 - Q_3 Q_4}{Q_1 Q_2 - Q_3^2}\right) \quad (31)$$

By substituting (31) into (30), gives w_1 as;

$$w_1 = \left(\frac{Q_2 Q_4 - Q_3 Q_5}{Q_1 Q_2 - Q_3^2}\right) \quad (32)$$

The MSE of $\boldsymbol{T_{DS}}$ in (21) is minimized for

$$\left.\begin{aligned} w_1 &= \left(\frac{Q_2 Q_4 - Q_3 Q_5}{Q_1 Q_2 - Q_3^2}\right) = w_{1F} \\ w_2 &= \left(\frac{Q_1 Q_5 - Q_3 Q_4}{Q_1 Q_2 - Q_3^2}\right) = w_{2F} \end{aligned}\right\} \quad (33)$$

By substituting (33) into (21) yields the optimum or minimum MSE of $\boldsymbol{T_{DS}}$ as;

$$MSE(T_{DS})_{opt} = \bar{Y}^2[1 + w_{1F}^2 Q_1 + w_{2F}^2 Q_2 + 2w_{1F}w_{2F}Q_3 - 2w_{1F}Q_4 - 2w_{2F}Q_5]$$

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$$MSE(T_{DS})_{opt} = \bar{Y}^2\left[1 + \left(\frac{Q_2Q_4-Q_3Q_5}{Q_1Q_2-Q_3^2}\right)^2 Q_1 + \left(\frac{Q_1Q_5-Q_3Q_4}{Q_1Q_2-Q_3^2}\right)^2 Q_2 + 2\left(\frac{Q_2Q_4-Q_3Q_5}{Q_1Q_2-Q_3^2}\right)\left(\frac{Q_1Q_5-Q_3Q_4}{Q_1Q_2-Q_3^2}\right)Q_3 - 2\left(\frac{Q_2Q_4-Q_3Q_5}{Q_1Q_2-Q_3^2}\right)Q_4 - 2\left(\frac{Q_1Q_5-Q_3Q_4}{Q_1Q_2-Q_3^2}\right)Q_5\right] \quad (34)$$

Let $Q_1 = k,\ Q_2 = h,\ Q_3 = c\ ,\ Q_4 = d$ and $Q_5 = l$

$$w_1 = \left(\frac{Q_2Q_4-Q_3Q_5}{Q_1Q_2-Q_3^2}\right)=\left(\frac{hd-cl}{kh-c^2}\right),\ w_2 = \left(\frac{Q_1Q_5-Q_3Q_4}{Q_1Q_2-Q_3^2}\right)=\left(\frac{kl-cd}{kh-c^2}\right)$$

$$w_1{}^2 = \left(\frac{h^2d^2+c^2l^2-2hdcl}{k^2h^2-2khc^2+c^4}\right),\ w_2{}^2 = \left(\frac{k^2l^2+c^2d^2-2klcd}{k^2h^2-2khc^2+c^4}\right)$$

Equation (34) can then be written as;

$$MSE(T_{DS})_{opt} = \bar{Y}^2\left[1 + \left(\frac{h^2d^2+c^2l^2-2hdcl}{k^2h^2-2khc^2+c^4}\right)k + \left(\frac{k^2l^2+c^2d^2-2klcd}{k^2h^2-2khc^2+c^4}\right)h + 2\left(\frac{hd-cl}{kh-c^2}\right)\left(\frac{kl-cd}{kh-c^2}\right)c - 2\left(\frac{kl-cd}{kh-c^2}\right)d - 2\left(\frac{hd-cl}{kh-c^2}\right)l\right]$$

$$MSE(T_{DS})_{opt} = \bar{Y}^2\left[\frac{kh-c^2+2cdl-hd^2-kl^2}{kh-c^2}\right] = \bar{Y}^2\left[\left(\frac{kh-c^2}{kh-c^2}\right)+\left(\frac{2cdl-hd^2-kl^2}{kh-c^2}\right)\right]$$

$$MSE(T_{DS})_{opt} \geq \bar{Y}^2\left[1+\frac{(2Q_3Q_4Q_5-Q_2Q_4^2-Q_1Q_5^2)}{(Q_1Q_2-Q_3^2)}\right] \quad (35)$$

The members of the proposed class of estimator T_{DS_i} of population mean $\bar{Y}$ of the study variable $\bar{y}$ were obtained by varying the values of the scalars that helps in designing the estimator T_{DS_i}. The scalars were uniquely varied in a manner that more brethren of ratio-cum-product estimators were obtained and presented in table 1.

Table 1: Some unique members of the proposed efficient class of estimator T_{DSj}

S/N	Estimators	Values of Scalars								
		w_1	w_2	a	b	ρ_{xz}	α_1	α_2	δ_1	δ_2
1	$\bar{y}$ the sample mean	1	0	0	0	0	0	0	0	0
2	$\bar{y}\left(\frac{\bar{X}+\rho_{xz}}{\bar{x}+\rho_{xz}}\right)\left(\frac{\bar{z}+\rho_{xz}}{\bar{Z}+\rho_{xz}}\right)$ Tailor (2005)	1	0	1	1	ρ_{xz}	1	1	0	0
3.	$\bar{y}\left(\frac{\bar{x}^*}{\bar{X}}\right)^{\delta_1}\left(\frac{\bar{Z}}{\bar{z}^*}\right)^{\delta_2}$ Sing et al (2011)	0	1	1	1	0	0	0	1	1
4	$\bar{y}\left(\frac{\bar{x}^*+\rho_{xz}}{\bar{X}+\rho_{xz}}\right)^{\delta_1}\left(\frac{\bar{Z}+\rho_{xz}}{\bar{z}^*+\rho_{xz}}\right)^{\delta_2}$ Vishwakarma et al (2014)	0	1	1	1	ρ_{xz}	0	0	1	1
5	$T_{DS1} = \bar{y}(w_1+w_2)$	w_1	w_2	1	1	0	0	0	0	0
6	$T_{DS2} = \bar{y}\left[w_1\left(\frac{\bar{X}}{\bar{x}}\right)+w_2\left(\frac{\bar{x}^*}{\bar{X}}\right)\right]$	w_1	w_2	1	1	0	1	0	1	0
7	$T_{DS3} = \bar{y}\left[w_1\left(\frac{\bar{z}}{\bar{Z}}\right)+w_2\left(\frac{\bar{Z}}{\bar{z}^*}\right)\right]$	w_1	w_2	1	1	0	0	1	0	1
8	$T_{DS4} = \bar{y}\left[w_1\left(\frac{\bar{X}+\rho_{xz}}{\bar{x}+\rho_{xz}}\right)+w_2\left(\frac{\bar{x}^*+\rho_{xz}}{\bar{X}+\rho_{xz}}\right)\right]$	w_1	w_2	1	1	ρ_{xz}	1	0	1	0
9	$T_{DS5} = \bar{y}\left[w_1\left(\frac{\bar{z}+\rho_{xz}}{\bar{Z}+\rho_{xz}}\right)+w_2\left(\frac{\bar{Z}+\rho_{xz}}{\bar{z}^*+\rho_{xz}}\right)\right]$	w_1	w_2	1	1	ρ_{xz}	0	1	0	1
10	$T_{DS6} = \bar{y}\left[w_1\left(\frac{a\bar{X}+\rho_{xz}}{a\bar{x}+\rho_{xz}}\right)+w_2\left(\frac{a\bar{x}^*+\rho_{xz}}{a\bar{X}+\rho_{xz}}\right)\right]$	w_1	w_2	a	1	ρ_{xz}	1	0	1	0
11	$T_{DS7} = \bar{y}\left[w_1\left(\frac{b\bar{z}+\rho_{xz}}{b\bar{Z}+\rho_{xz}}\right)+w_2\left(\frac{b\bar{Z}+\rho_{xz}}{b\bar{z}^*+\rho_{xz}}\right)\right]$	w_1	w_2	1	b	ρ_{xz}	0	1	0	1

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12	$T_{DS8}=\bar{y}\left[w_1\left(\frac{a\bar{X}+\rho_{xz}}{a\bar{x}+\rho_{xz}}\right)^{\alpha_1}+w_2\left(\frac{a\bar{x}^*+\rho_{xz}}{a\bar{X}+\rho_{xz}}\right)^{\delta_1}\right]$	w_1	w_2	a	1	ρ_{xz}	α_1	0	δ_1	0
13	$T_{DS9}=\bar{y}\left[w_1\left(\frac{b\bar{z}+\rho_{xz}}{b\bar{Z}+\rho_{xz}}\right)^{\alpha_2}+w_2\left(\frac{b\bar{Z}+\rho_{xz}}{b\bar{z}^*+\rho_{xz}}\right)^{\delta_2}\right]$	w_1	w_2	1	b	ρ_{xz}	0	α_2	0	δ_2
14	$T_{DS10}=\bar{y}\left[w_1\left(\frac{\bar{X}+\rho_{xz}}{\bar{x}+\rho_{xz}}\right)\left(\frac{\bar{z}+\rho_{xz}}{\bar{Z}+\rho_{xz}}\right)+w_2\left(\frac{\bar{x}^*+\rho_{xz}}{\bar{X}+\rho_{xz}}\right)\left(\frac{\bar{Z}+\rho_{xz}}{\bar{z}^*+\rho_{xz}}\right)\right]$	w_1	w_2	1	1	ρ_{xz}	1	1	1	1
15	$T_{DS11}=\bar{y}\left[w_1\left(\frac{a\bar{X}+\rho_{xz}}{a\bar{x}+\rho_{xz}}\right)\left(\frac{bz+\rho_{xz}}{b\bar{Z}+\rho_{xz}}\right)+w_2\left(\frac{a\bar{x}^*+\rho_{xz}}{a\bar{X}+\rho_{xz}}\right)\left(\frac{bz+\rho_{xz}}{b\bar{Z}+\rho_{xz}}\right)\right]$	w_1	w_2	a	b	ρ_{xz}	1	1	1	1
16	$T_{DS12}=\bar{y}\left[w_1\left(\frac{a\bar{X}+\rho_{xz}}{a\bar{x}+\rho_{xz}}\right)^{\alpha_1}\left(\frac{bz+\rho_{xz}}{b\bar{Z}+\rho_{xz}}\right)^{\alpha_2}+w_2\left(\frac{a\bar{x}^*+\rho_{xz}}{a\bar{X}+\rho_{xz}}\right)^{\delta_1}\left(\frac{b\bar{Z}+\rho_{xz}}{b\bar{z}^*+\rho_{xz}}\right)^{\delta_2}\right]$	w_1	w_2	a	b	ρ_{xz}	α_1	α_2	δ_1	δ_2

Table 2: Members of T_{DSj} , $j = 1, 2, \dots 16$ with their Biases case I

S/N	T_{DSj}	MSE
1	T_{DS1}	Unbiased
2	T_{DS2}	$\theta(\bar{Y})\left(\lambda_1^2C_x^2-\lambda_1C_{yx}+\lambda_2C_{yz}-\lambda_1\lambda_2C_{xz}\right)$
3	T_{DS3}	$\theta(\bar{Y})\left(\frac{\delta_1(\delta_1-1)}{2}{g_1}^2C_x^2-\frac{\delta_2(\delta_2+1)}{2}{g_2}^2C_z^2+g_1\delta_1C_{yx}-g_2\delta_2C_{yz}-g_1g_2\delta_1\delta_2C_{xz}\right)$
4	T_{DS4}	$\theta(\bar{Y})\left(\frac{\delta_1(\delta_1-1)}{2}{g_1}^2\lambda_1^2C_x^2-\frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_2^2C_z^2+g_1\delta_1\lambda_1C_{yx}-g_2\delta_2\lambda_2C_{yz}-g_1g_2\delta_1\delta_2\lambda_1\lambda_2C_{xz}\right)$
5	T_{DS5}	$\theta(\bar{Y})\left[w_1+w_2-\frac{1}{\theta}\right]$
6	T_{DS6}	$\theta(\bar{Y})\left[w_1\left(\frac{1}{\theta}+\lambda_1^2C_x^2\right)+w_2\left(\frac{1}{\theta}+g_1\lambda_1C_{yx}\right)-\frac{1}{\theta}\right]$
7	T_{DS7}	$\theta(\bar{Y})\left[w_1\left(\frac{1}{\theta}+\lambda_2C_{yz}\right)+w_2\left(\frac{1}{\theta}-g_2\lambda_2C_{yz}-{g_2}^2\lambda_2^2C_z^2\right)-\frac{1}{\theta}\right]$
8	T_{DS8}	$\theta(\bar{Y})\left[w_1\left(\frac{1}{\theta}+\lambda_1^2C_x^2\right)+w_2\left(\frac{1}{\theta}+g_1\lambda_1C_{yx}\right)-\frac{1}{\theta}\right]$
9	T_{DS9}	$\theta(\bar{Y})\left[w_1\left(\frac{1}{\theta}+\lambda_2C_{yz}\right)+w_2\left(\frac{1}{\theta}-g_2\lambda_2C_{yz}-{g_2}^2\lambda_2^2C_z^2\right)-\frac{1}{\theta}\right]$
10	T_{DS10}	$\theta(\bar{Y})\left[w_1\left(\frac{1}{\theta}+\lambda_a^2C_x^2\right)+w_2\left(\frac{1}{\theta}-g_1\lambda_aC_{yx}\right)-\frac{1}{\theta}\right]$
11	T_{DS11}	$\theta(\bar{Y})\left[w_1\left(\frac{1}{\theta}+\lambda_bC_{yz}\right)+w_2\left(\frac{1}{\theta}-g_2\lambda_bC_{yz}-{g_2}^2\lambda_b^2C_z^2\right)-\frac{1}{\theta}\right]$
12	T_{DS12}	$\theta(\bar{Y})\left[w_1\left(\frac{1}{\theta}+\frac{\alpha_1(\alpha_1+1)}{2}\lambda_a^2C_x^2\right)+w_2\left(\frac{1}{\theta}+g_1\delta_1\lambda_aC_{yx}+\frac{\delta_1(\delta_1-1)}{2}{g_1}^2\lambda_a^2C_x^2\right)-\frac{1}{\theta}\right]$
13	T_{DS13}	$\theta(\bar{Y})\left[w_1\left(\frac{1}{\theta}+\alpha_2\lambda_bC_{yz}+\frac{\alpha_2(\alpha_2-1)}{2}\lambda_b^2C_z^2\right)+w_2\left(\frac{1}{\theta}-g_2\delta_2\lambda_bC_{yz}-\frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_b^2C_z^2\right)-\frac{1}{\theta}\right]$
14	T_{DS14}	$\theta(\bar{Y})\left[w_1\left(\frac{1}{\theta}+\lambda_2C_{yz}-\left(C_{xz}+C_{yx}\right)\lambda_1\lambda_2+\lambda_1^2C_x^2\right)+w_2\left(\frac{1}{\theta}-g_2\lambda_2C_{yz}-{g_2}^2\lambda_2^2C_z^2++g_1\lambda_1C_{yx}-g_1g_2\delta_1\delta_2\lambda_1\lambda_2C_{xz}\right)-\frac{1}{\theta}\right]$
15	T_{DS15}	$\theta(\bar{Y})\left[w_1\left(\frac{1}{\theta}+\lambda_bC_{yz}-\left(C_{xz}+C_{yx}\right)\lambda_a\lambda_b+\lambda_a^2C_x^2\right)+w_2\left(\frac{1}{\theta}-g_2\lambda_bC_{yz}-{g_2}^2\lambda_b^2C_z^2++g_1\lambda_aC_{yx}-g_1g_2\delta_1\delta_2\lambda_a\lambda_bC_{xz}\right)-\frac{1}{\theta}\right]$

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16	T_{DS16}	$\theta\,\bar{Y}[w_1\left(\frac{1}{\theta}+\alpha_2\lambda_b C_{yz}+\frac{\alpha_2(\alpha_2-1)}{2}\lambda_b^2 C_z^2-\alpha_1\alpha_2\lambda_a\lambda_b C_{xz}-\alpha_1\lambda_a\lambda_b C_{yx}+\frac{\alpha_1(\alpha_1+1)}{2}\lambda_a^2 C_x^2\right)+ w_2\left(\frac{1}{\theta}-g_2\delta_2\lambda_b C_{yz}-\frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_b^2 C_z^2+g_1\delta_1\lambda_a C_{yx}-g_1g_2\delta_1\delta_2\lambda_a\lambda_b C_{xz}+ \frac{\delta_1(\delta_1-1)}{2}{g_1}^2\lambda_a^2 C_x^2\right)-\frac{1}{\theta}]$

4.1.2 The members of the ratio-cum-product estimators T_{Dsj}, j=1, 2 …16with their MSE's case I

The MSEs of the collection of ratio-cum-product estimators $\left(T_{Dsj}\right)$,

$j=1,2,\ldots 16$ i.e,T_{DS1}, T_{DS2}, T_{DS3}, T_{DS4}, T_{DS5}, T_{DS6}, T_{DS7}, T_{DS8}, T_{DS9}, T_{DS10}, T_{DS11}, T_{DS12}, T_{DS13}, T_{DS14}, T_{DS15}, T_{DS16} were derived and presented as follows.

$$T_{DS1}=\theta(\bar{Y})^2{c_y}^2 \quad (36)$$

$$T_{DS2}=\theta(\bar{Y})^2({c_y}^2+3\lambda_1^2C_x^2+3\lambda_2^2C_z^2+2\lambda_1C_{yx}+2\lambda_2C_{yz}-2\lambda_1\lambda_2C_{xz}) \quad (37)$$

$$T_{DS3}=\theta(\bar{Y})^2\left(C_y^2+{\delta_1}^2{g_1}^2C_x^2+{\delta_2}^2{g_2}^2C_z^2+2g_1\delta_1C_{xy}-2g_2\delta_2C_{yz}-2g_1g_2\delta_1\delta_2C_{xz}\right) \quad (38)$$

$$T_{DS4}=\theta(\bar{Y})^2\left(C_y^2+{\delta_1}^2{g_1}^2\lambda_1^2c_x^2+{\delta_2}^2{g_2}^2\lambda_2^2C_z^2+2g_1\delta_1\lambda_1C_{xy}-2g_2\delta_2\lambda_2C_{yz}-2g_1g_2\delta_1\delta_2\lambda_1\lambda_2C_{xz}\right) \quad (39)$$

$$T_{DS5}=\theta(\bar{Y})^2\left[({w_1}^2+{w_2}^2+2w_1w_2)\left(\frac{1}{\theta}+{c_y}^2\right)-(2w_1+2w_2+1)\left(\frac{1}{\theta}\right)\right] \quad (40)$$

$$T_{DS6}=\theta(\bar{Y})^2\left[{w_1}^2\left(\frac{1}{\theta}+{c_y}^2+3\lambda_1^2C_x^2-4\lambda_1C_{yx}\right)+{w_2}^2\left(\frac{1}{\theta}+{c_y}^2+{g_1}^2\lambda_1^2C_x^2+4g_1\lambda_1C_{yx}\right)+2w_1w_2\left(\frac{1}{\theta}+{c_y}^2+(1-g_1)\lambda_1^2C_x^2-2\lambda_1C_{yx}+2g_1\lambda_1C_{yx}\right)-2w_1\left(\frac{1}{\theta}+\lambda_1^2C_x^2-\lambda_1C_{yx}\right)-2w_2\left(\frac{1}{\theta}+g_1\lambda_1C_{yx}\right)+\frac{1}{\theta}\right] \quad (41)$$

$$T_{DS7}=\theta(\bar{Y})^2\left[{w_1}^2\left(\frac{1}{\theta}+{c_y}^2+4\lambda_1C_{yz}\right)+{w_2}^2\left(\frac{1}{\theta}+{c_y}^2+3{g_2}^2\lambda_b^2C_z^2-4g_2\lambda_2C_{yz}\right)+2w_1w_2\left(\frac{1}{\theta}+{c_y}^2+2\lambda_2C_{yz}\right)-2w_1\left(\frac{1}{\theta}+\lambda_2C_{yz}\right)-2w_2\left(\frac{1}{\theta}-{g_2}^2\lambda_2^2C_z^2-g_2\lambda_2C_{yz}\right)+\frac{1}{\theta}\right] \quad (42)$$

$$T_{DS8}=\theta(\bar{Y})^2\left[{w_1}^2\left(\frac{1}{\theta}+{c_y}^2+3\lambda_1^2C_x^2-4\lambda_1C_{yx}\right)+{w_2}^2\left(\frac{1}{\theta}+{c_y}^2+{g_1}^2\lambda_1^2C_x^2+4g_1\lambda_1C_{yx}\right)+2w_1w_2\left(\frac{1}{\theta}+{c_y}^2+(1-g_1)\lambda_1^2C_x^2-2\lambda_1C_{yx}+2g_1\lambda_1C_{yx}\right)-2w_1\left(\frac{1}{\theta}+\lambda_1^2C_x^2-\lambda_1C_{yx}\right)-2w_2\left(\frac{1}{\theta}+g_1\lambda_1C_{yx}\right)+\frac{1}{\theta}\right] \quad (43)$$

$$T_{DS9}=\theta(\bar{Y})^2\left[{w_1}^2\left(\frac{1}{\theta}+{c_y}^2+\lambda_2^2C_z^2+4\lambda_2C_{yz}\right)+{w_2}^2\left(\frac{1}{\theta}+{c_y}^2+3{g_2}^2\lambda_2^2C_z^2-4g_2\lambda_2C_{yz}\right)+2w_1w_2\left(\frac{1}{\theta}+{c_y}^2+{g_2}^2\lambda_2^2C_z^2+2\lambda_2C_{yz}\right)-2w_1\left(\frac{1}{\theta}+\lambda_2C_{yz}\right)-2w_2\left(\frac{1}{\theta}+{g_2}^2\lambda_2^2C_z^2-g_2\lambda_2C_{yz}\right)+\frac{1}{\theta}\right] \quad (44)$$

$$T_{DS10}=\theta(\bar{Y})^2\left[{w_1}^2\left(\frac{1}{\theta}+{c_y}^2+3\lambda_a^2C_x^2-4\lambda_aC_{yx}\right)+{w_2}^2\left(\frac{1}{\theta}+{c_y}^2+{g_1}^2\lambda_a^2C_x^2+4g_1\lambda_aC_{yx}\right)+2w_1w_2\left(\frac{1}{\theta}+{c_y}^2+(1-g_1)\lambda_a^2C_x^2-2\lambda_aC_{yx}+2g_1\lambda_aC_{yx}\right)-2w_1\left(\frac{1}{\theta}+\lambda_a^2C_x^2-\lambda_aC_{yx}\right)-2w_2\left(\frac{1}{\theta}+g_1\lambda_aC_{yx}\right)+\frac{1}{\theta}\right] \quad (45)$$

$$T_{DS11}=\theta(\bar{Y})^2\left[{w_1}^2\left(\frac{1}{\theta}+{c_y}^2+\lambda_b^2C_z^2+4\lambda_bC_{yz}\right)+{w_2}^2\left(\frac{1}{\theta}+{c_y}^2+3{g_2}^2\lambda_a^2C_z^2-4g_2\lambda_bC_{yz}\right)+2w_1w_2\left(\frac{1}{\theta}+{c_y}^2+{g_2}^2\lambda_b^2C_z^2+2\lambda_bC_{yz}\right)-2w_1\left(\frac{1}{\theta}+\lambda_bC_{yz}\right)-2w_2\left(\frac{1}{\theta}+{g_2}^2\lambda_b^2C_z^2-g_2\lambda_bC_{yz}\right)+\frac{1}{\theta}\right] \quad (46)$$

$$T_{DS12}=\theta(\bar{Y})^2\left[{w_1}^2\left(\frac{1}{\theta}+{c_y}^2+\alpha_1(2\alpha_1+1)\lambda_a^2c_x^2-4\alpha_1\lambda_aC_{yx}\right)+{w_2}^2\left(\frac{1}{\theta}+{c_y}^2+\delta_1(2\delta_1-1){g_1}^2\lambda_a^2c_x^2+4g_1\delta_1\lambda_1C_{yx}\right)+2w_1w_2\left(\frac{1}{\theta}+{c_y}^2+\frac{\delta_1(\delta_1-1)}{2}{g_1}^2\lambda_a^2c_x^2+\frac{\alpha_1(\alpha_1+1)}{2}-\alpha_1g_1\delta_1)\lambda_a^2c_x^2-2\alpha_1\lambda_1C_{yx}++2g_1\delta_1\lambda_aC_{yx}\right)-2w_1\left(\frac{1}{\theta}+\frac{\alpha_1(\alpha_1+1)}{2}\lambda_a^2c_x^2-\alpha_1\lambda_aC_{yx}\right)-2w_2\left(\frac{1}{\theta}+\frac{\delta_1(\delta_1-1)}{2}{g_2}^2\lambda_a^2c_x^2+g_1\delta_1\lambda_aC_{yx}\right)+\frac{1}{\theta}\right] \quad (47)$$

$$T_{DS13}=\theta(\bar{Y})^2\left[{w_1}^2\left(\frac{1}{\theta}+{c_y}^2+\alpha_2(2\alpha_2-1)\lambda_b^2c_z^2+4\alpha_2\lambda_bC_{yz}\right)+{w_2}^2\left(\frac{1}{\theta}+{c_y}^2+\delta_2(2\delta_2+1){g_2}^2\lambda_b^2c_z^2-4g_2\delta_2\lambda_2C_{yz}\right)+2w_1w_2\left(\frac{1}{\theta}+{c_y}^2+\left(\frac{\alpha_2(\alpha_2-1)}{2}+\alpha_2g_2\delta_2\right)\lambda_b^2c_z^2-\frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_b^2c_z^2+2\alpha_2\lambda_2C_{yz}\right)-2w_1\left(\frac{1}{\theta}+\frac{\alpha_2(\alpha_2-1)}{2}\lambda_b^2c_z^2+\alpha_2\lambda_bC_{yz}\right)-2w_2\left(\frac{1}{\theta}-\frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_b^2c_z^2-g_2\delta_2\lambda_bC_{yz}\right)+\frac{1}{\theta}\right] \quad (48)$$

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$\boldsymbol{T_{DS14}} = T_{DS4} = \theta(\bar{Y})^2\Big[{w_1}^2\left(\frac{1}{\theta}+{c_y}^2+3\lambda_1^2C_x^2+\lambda_2^2c_z^2-4\lambda_1C_{yx}+4\lambda_2C_{yz}-4\lambda_1\lambda_2\,C_{xz}\right)+{w_2}^2\left(\frac{1}{\theta}+{c_y}^2+{g_1}^2\lambda_1^2C_x^2+3{g_2}^2\lambda_2^2c_z^2+4g_1\lambda_1C_{yx}-4g_2\lambda_2C_{yz}-4g_1g_2\lambda_1\lambda_2C_{xz}\right)+2w_1w_2\left(\frac{1}{\theta}+{c_y}^2-{g_1}^2\lambda_1^2c_x^2+{g_2}^2\lambda_2^2c_z^2+(1-g_1)\lambda_1^2C_x^2+(g_1-g_2)\lambda_1\lambda_2C_{yz}+(g_1g_2+1)\lambda_1\lambda_2C_{xz}-2\lambda_1C_{yx}+2\lambda_2C_{yz}+2g_1\lambda_1C_{yx}\right)-2w_1\left(\frac{1}{\theta}+\lambda_1^2C_x^2-\lambda_1C_{yx}+\lambda_2C_{yz}-\lambda_1\lambda_2C_{xz}\right)-2w_2\left(\frac{1}{\theta}-{g_2}^2\lambda_2^2c_z^2+g_1\lambda_1C_{yx}-g_2\lambda_2C_{yz}-g_1g_2\lambda_a\lambda_bC_{xz}\right)+\frac{1}{\theta}\Big]$ (49)

$\boldsymbol{T_{DS15}} = \theta(\bar{Y})^2\Big[{w_1}^2\left(\frac{1}{\theta}+{c_y}^2+3\lambda_a^2C_x^2+\lambda_b^2c_z^2-4\lambda_1C_{yx}+4\lambda_bC_{yz}-4\lambda_a\lambda_b\,C_{xz}\right)+{w_2}^2\left(\frac{1}{\theta}+{c_y}^2+{g_1}^2\lambda_a^2C_x^2+3{g_2}^2\lambda_b^2c_z^2+4g_1\lambda_aC_{yx}-4g_2\lambda_bC_{yz}-4g_1g_2\lambda_a\lambda_bC_{xz}\right)+2w_1w_2\left(\frac{1}{\theta}+{c_y}^2-{g_1}^2\lambda_a^2c_x^2+{g_2}^2\lambda_b^2c_z^2+(1-g_1)\lambda_a^2C_x^2+(g_1-g_2)\lambda_a\lambda_bC_{yz}+(g_1g_2+1)\lambda_a\lambda_bC_{xz}-2\lambda_aC_{yx}+2\lambda_bC_{yz}+2g_1\lambda_aC_{yx}\right)-2w_1\left(\frac{1}{\theta}+\lambda_a^2C_x^2-\lambda_aC_{yx}+\lambda_bC_{yz}-\lambda_a\lambda_bC_{xz}\right)-2w_2\left(\frac{1}{\theta}-{g_2}^2\lambda_b^2c_z^2+g_1\lambda_aC_{yx}-g_2\lambda_bC_{yz}-g_1g_2\lambda_a\lambda_bC_{xz}\right)+\frac{1}{\theta}\Big]$ (50)

$\boldsymbol{T_{DS16}}$: The MSE of $\boldsymbol{T_{DS16}}$ is that of the mother estimator and it is as presented in equation (21)

Case II

In the case that the second sample of size m_2 was selected separately from the preliminary one, then the suggested estimator would still be same, except that the bias, relative bias and MSE in this case would be dissimilar from that of Case I. The bias and MSE in this case were derived by setting

$$\mathrm{E}(e_xe_z) = E\left(e_ye_z\right) = 0$$

Invariably;

$$\rho_{xz}C_xC_z = \rho_{xz}C_yC_z = C_{xz} = C_{yz} = 0$$

Thus for case II of the suggested estimator, the bias and Mean Square Error were obtained and presented as follows:

$$\boldsymbol{E(T_{DS} - \bar{Y})} = \theta'\,\overline{Y}\left[w_1\left(\frac{1}{\theta'}+\frac{\alpha_2(\alpha_2-1)}{2}\lambda_b^2C_z^2-\alpha_1\lambda_a\lambda_bC_{yx}+\frac{\alpha_1(\alpha_1+1)}{2}\lambda_a^2C_x^2\right)+w_2\left(\frac{1}{\theta'}-\frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_b^2C_z^2+g_1\delta_1\lambda_aC_{yx}+\frac{\delta_1(\delta_1-1)}{2}{g_1}^2\lambda_a^2C_x^2\right)-\frac{1}{\theta'}\right] \quad (51)$$

$$\frac{E(T_{DS} - \bar{Y})}{\bar{Y}} = \theta'\left[w_1\left(\frac{1}{\theta'}+\frac{\alpha_2(\alpha_2-1)}{2}\lambda_b^2C_z^2-\alpha_1\lambda_a\lambda_bC_{yx}+\frac{\alpha_1(\alpha_1+1)}{2}\lambda_a^2C_x^2\right)+w_2\left(\frac{1}{\theta'}-\frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_b^2C_z^2+g_1\delta_1\lambda_aC_{yx}+\frac{\delta_1(\delta_1-1)}{2}{g_1}^2\lambda_a^2C_x^2\right)-\frac{1}{\theta'}\right] \quad (52)$$

$$MSE(T_{DS}) = E(T_{DS} - \bar{Y})^2 = \bar{Y}^2[1+w_1^2(U_1)+w_2^2(U_2)+2W_1W_2(U_3)-2W_1(U_4)-2W_2(U_5)] \quad (53)$$

Where,

$$\boldsymbol{U_1} = \left(1+\theta'[{c_y}^2+\alpha_1(2\alpha_1+1)\lambda_a^2c_x^2+\alpha_2(2\alpha_2-1)\lambda_b^2c_z^2-4\alpha_1\lambda_aC_{yx}]\right) \quad (54)$$

$$\boldsymbol{U_2} = \left(\boldsymbol{1}+\theta'[{c_y}^2+\delta_1(2\delta_1-1){g_1}^2\lambda_a^2c_x^2+\delta_2(2\delta_2+1){g_2}^2\lambda_b^2c_z^2+4g_1\delta_1\lambda_aC_{yx}]\right) \quad (55)$$

$$\boldsymbol{U_3} = \left(\boldsymbol{1}+\theta'\left[{c_y}^2+\frac{\delta_1(\delta_1-1)}{2}{g_1}^2\lambda_a^2c_x^2-\frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_b^2c_z^2+\left(\frac{\alpha_2(\alpha_2-1)}{2}+\alpha_2g_2\delta_2\right)\lambda_b^2c_z^2+\left(\frac{\alpha_1(\alpha_1+1)}{2}-\alpha_1g_1\delta_1\right)\lambda_a^2c_x^2-2\alpha_1\lambda_aC_{yx}+2g_1\delta_1\lambda_aC_{yx}\right)\right]\right) \quad (56)$$

$$\boldsymbol{U_4} = \left(1+\theta'\left[\frac{\alpha_1(\alpha_1+1)}{2}\lambda_a^2c_x^2+\frac{\alpha_2(\alpha_2-1)}{2}\lambda_b^2c_z^2-\alpha_1\lambda_aC_{yx}\right]\right) \quad (57)$$

$$\boldsymbol{U_5} = \left(1+\theta'\left[\frac{\delta_1(\delta_1-1)}{2}{g_2}^2\lambda_a^2c_x^2-\frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_b^2c_z^2+g_1\delta_1\lambda_aC_{yx}\right]\right) \quad (58)$$

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Similarly, the Optimal Mean Square Error for case II was obtained and presented as

$$MSE(T^*_{DS})_{opt} = \bar{Y}^2[1 + w_{1S}^2U_1 + w_{2S}^2U_2 + 2w_{1S}w_{2S}U_3 - 2w_{1F}U_4 - 2w_{2F}U_5] \quad (59)$$

$$MSE(T^*_{DS})_{opt} \geq \bar{Y}^2\left[1 + \frac{(2U_3U_4U_5 - U_2U_4^2 - U_1U_5^2)}{(U_1U_2 - U_3^2)}\right] \quad (60)$$

Where

$$\left.\begin{aligned} w_1 = w_{1S} &= \left(\frac{U_2U_4 - U_3U_5}{U_1U_2 - U_3^2}\right) \\ w_2 = w_{2S} &= \left(\frac{U_1U_5 - U_3U_4}{U_1U_2 - U_3^2}\right) \end{aligned}\right\} \quad (61)$$

Table 3: Members of T_{DSj}, $j = 1, 2, \dots 16$ with their Biases case II

S/N	T^*_{DSj}	MSE
1	T^*_{DS1}	Unbiased
2	T^*_{DS2}	$\theta(\bar{Y})(\lambda_1^2C_x^2 - \lambda_1C_{yx})$
3	T^*_{DS3}	$\theta(\bar{Y})\left(\frac{\delta_1(\delta_1-1)}{2}g_1{}^2C_x^2 - \frac{\delta_2(\delta_2+1)}{2}g_2{}^2C_z^2 + g_1\delta_1C_{yx}\right)$
4	T^*_{DS4}	$\theta(\bar{Y})\left(\frac{\delta_1(\delta_1-1)}{2}g_1{}^2\lambda_1^2C_x^2 - \frac{\delta_2(\delta_2+1)}{2}g_2{}^2\lambda_2^2C_z^2 + g_1\delta_1\lambda_1C_{yx}\right)$
5	T^*_{DS5}	$\theta'(\bar{Y})\left[w_1 + w_2 - \frac{1}{\theta'}\right]$
6	T^*_{DS6}	$\theta'(\bar{Y})\left[w_1\left(\frac{1}{\theta'} + \lambda_1^2C_x^2\right) + w_2\left(\frac{1}{\theta'} + g_1\lambda_1C_{yx}\right) - \frac{1}{\theta'}\right]$
7	T^*_{DS7}	$\theta'(\bar{Y})\left[w_1\left(\frac{1}{\theta'} + \lambda_2C_{yz}\right) + w_2\left(\frac{1}{\theta'} - g_2\lambda_2C_{yz} - g_2{}^2\lambda_2^2C_z^2\right) - \frac{1}{\theta'}\right]$
8	T^*_{DS8}	$\theta'(\bar{Y})\left[w_1\left(\frac{1}{\theta'} + \lambda_1^2C_x^2\right) + w_2\left(\frac{1}{\theta'} + g_1\lambda_1C_{yx}\right) - \frac{1}{\theta'}\right]$
9	T^*_{DS9}	$\theta'(\bar{Y})\left[w_1\left(\frac{1}{\theta'} + \lambda_2C_{yz}\right) + w_2\left(\frac{1}{\theta'} - g_2\lambda_2C_{yz} - g_2{}^2\lambda_2^2C_z^2\right) - \frac{1}{\theta'}\right]$
10	T^*_{DS10}	$\theta'(\bar{Y})\left[w_1\left(\frac{1}{\theta'} + \lambda_a^2C_x^2\right) + w_2\left(\frac{1}{\theta'} - g_1\lambda_aC_{yx}\right) - \frac{1}{\theta'}\right]$
11	T^*_{DS11}	$\theta'(\bar{Y})\left[w_1\left(\frac{1}{\theta'}\right) + w_2\left(\frac{1}{\theta'} - g_2{}^2\lambda_b^2C_z^2\right) - \frac{1}{\theta'}\right]$
12	T^*_{DS12}	$\theta'(\bar{Y})\left[w_1\left(\frac{1}{\theta'} + \frac{\alpha_1(\alpha_1+1)}{2}\lambda_a^2C_x^2\right) + w_2\left(\frac{1}{\theta'} + g_1\delta_1\lambda_aC_{yx} + \frac{\delta_1(\delta_1-1)}{2}g_1{}^2\lambda_a^2C_x^2\right) - \frac{1}{\theta'}\right]$
13	T^*_{DS13}	$\theta'(\bar{Y})\left[w_1\left(\frac{1}{\theta'} + \frac{\alpha_2(\alpha_2-1)}{2}\lambda_b^2C_z^2\right) + w_2\left(\frac{1}{\theta'} - \frac{\delta_2(\delta_2+1)}{2}g_2{}^2\lambda_b^2C_z^2\right) - \frac{1}{\theta'}\right]$
14	T^*_{DS14}	$\theta'(\bar{Y})\left[w_1\left(\frac{1}{\theta'} - \lambda_1\lambda_2C_{yx} + \lambda_1^2C_x^2\right) + w_2\left(\frac{1}{\theta'} - g_2{}^2\lambda_2^2C_z^2 + +g_1\lambda_1C_{yx}\right) - \frac{1}{\theta'}\right]$
15	T^*_{DS15}	$\theta'(\bar{Y})\left[w_1\left(\frac{1}{\theta'} - \lambda_a\lambda_bC_{yx} + \lambda_a^2C_x^2\right) + w_2\left(\frac{1}{\theta'} - g_2{}^2\lambda_b^2C_z^2 + +g_1\lambda_aC_{yx}\right) - \frac{1}{\theta'}\right]$
16	T^*_{DS16}	$\theta'\bar{Y}[w_1\left(\frac{1}{\theta'} + \frac{\alpha_2(\alpha_2-1)}{2}\lambda_b^2C_z^2 - \alpha_1\lambda_a\lambda_bC_{yx} + \frac{\alpha_1(\alpha_1+1)}{2}\lambda_a^2C_x^2\right) + w_2\left(\frac{1}{\theta'} - \frac{\delta_2(\delta_2+1)}{2}g_2{}^2\lambda_b^2C_z^2 + g_1\delta_1\lambda_aC_{yx} + \frac{\delta_1(\delta_1-1)}{2}g_1{}^2\lambda_a^2C_x^2\right) - \frac{1}{\theta'}]$

4.1.3 The members of the ratio-cum-product estimators T_{Dsj}, j=1, 2 …16with their MSEs, case II

The MSEs of the collection of ratio-cum-product estimators (even case)(T^*_{Dsj}),

$j = 1, 2, \dots 16$ i.e T^*_{DS1}, T^*_{DS2}, T^*_{DS3}, T^*_{DS4}, T^*_{DS5}, T^*_{DS6}, T^*_{DS7}, T^*_{DS8}, T^*_{DS9}, T^*_{DS10}, T^*_{DS11}, T^*_{DS12}, T^*_{DS13}, T^*_{DS14}, T^*_{DS15} and T^*_{DS16} were derived and presented as follows.

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$\boldsymbol{T^*}_{\boldsymbol{DS1}} = \theta'(\bar{Y})^2 {c_y}^2$ (62)

$\boldsymbol{T^*}_{\boldsymbol{DS2}} = \theta'(\bar{Y})^2({c_y}^2 + 3\lambda_1^2 C_x^2 + 3\lambda_2^2 C_z^2 + 2\lambda_1 C_{yx}$ (63)

$\boldsymbol{T^*}_{\boldsymbol{DS3}} = \theta'(\bar{Y})^2\left(C_y^2 + {\delta_1}^2 {g_1}^2 C_x^2 + {\delta_2}^2 {g_2}^2 C_z^2 + 2g_1\delta_1 C_{xy}\right)$ (64)

$\boldsymbol{T^*}_{\boldsymbol{DS4}} = \theta'(\bar{Y})^2\left(C_y^2 + {\delta_1}^2 {g_1}^2 \lambda_1^2 c_x^2 + {\delta_2}^2 {g_2}^2 \lambda_2^2 C_z^2 + 2g_1\delta_1\lambda_1 C_{xy}\right)$ (65)

$\boldsymbol{T^*}_{\boldsymbol{DS5}} = \theta'(\bar{Y})^2\left[({w_1}^2 + {w_2}^2 + 2w_1w_2)\left(\frac{1}{\theta'} + {c_y}^2\right) - (2w_1 + 2w_2 + 1)\left(\frac{1}{\theta'}\right)\right]$ (66)

$\boldsymbol{T^*}_{\boldsymbol{DS6}} = \theta'(\bar{Y})^2\Big[{w_1}^2\left(\frac{1}{\theta'} + {c_y}^2 + 3\lambda_1^2 C_x^2 - 4\lambda_1 C_{yx}\right) + {w_2}^2\left(\frac{1}{\theta} + {c_y}^2 + {g_1}^2\lambda_1^2 C_x^2 + 4g_1\lambda_1 C_{yx}\right) + 2w_1w_2\Big(\frac{1}{\theta'} + {c_y}^2 +$
$(1-g_1)\lambda_1^2 C_x^2 - 2\lambda_1 C_{yx} + 2g_1\lambda_1 C_{yx}\Big) - 2w_1\left(\frac{1}{\theta} + \lambda_1^2 C_x^2 - \lambda_1 C_{yx}\right) - 2w_2\left(\frac{1}{\theta} + g_1\lambda_1 C_{yx}\right) + \frac{1}{\theta'}\Big]$ (67)

$\boldsymbol{T^*}_{\boldsymbol{DS7}} == \theta'(\bar{Y})^2\left[{w_1}^2\left(\frac{1}{\theta'} + {c_y}^2\right) + {w_2}^2\left(\frac{1}{\theta'} + {c_y}^2 + 3{g_2}^2\lambda_b^2 C_z^2\right) + 2w_1w_2\left(\frac{1}{\theta} + {c_y}^2\right) - 2w_1\left(\frac{1}{\theta'}\right) - 2w_2\left(\frac{1}{\theta} - {g_2}^2\lambda_2^2 C_z^2\right) + \frac{1}{\theta'}\right]$
(68)

$\boldsymbol{T^*}_{\boldsymbol{DS8}} = \theta'(\bar{Y})^2\Big[{w_1}^2\left(\frac{1}{\theta'} + {c_y}^2 + 3\lambda_1^2 C_x^2 - 4\lambda_1 C_{yx}\right) + {w_2}^2\left(\frac{1}{\theta'} + {c_y}^2 + {g_1}^2\lambda_1^2 C_x^2 + 4g_1\lambda_1 C_{yx}\right) + 2w_1w_2\Big(\frac{1}{\theta'} + {c_y}^2 +$
$(1-g_1)\lambda_1^2 C_x^2 - 2\lambda_1 C_{yx} + 2g_1\lambda_1 C_{yx}\Big) - 2w_1\left(\frac{1}{\theta'} + \lambda_1^2 C_x^2 - \lambda_1 C_{yx}\right) - 2w_2\left(\frac{1}{\theta} + g_1\lambda_1 C_{yx}\right) + \frac{1}{\theta'}\Big]$ (69)

$\boldsymbol{T^*}_{\boldsymbol{DS9}} = \theta'(\bar{Y})^2\Big[{w_1}^2\left(\frac{1}{\theta'} + {c_y}^2 + \lambda_2^2 C_z^2\right) + {w_2}^2\left(\frac{1}{\theta'} + {c_y}^2 + 3{g_2}^2\lambda_2^2 C_z^2\right) + 2w_1w_2\left(\frac{1}{\theta'} + {c_y}^2 + {g_2}^2\lambda_2^2 C_z^2\right) - 2w_1\left(\frac{1}{\theta'}\right) -$
$2w_2\left(\frac{1}{\theta'} + {g_2}^2\lambda_2^2 C_z^2\right) + \frac{1}{\theta'}\Big]$ (70)

$\boldsymbol{T^*}_{\boldsymbol{DS10}} = \theta'(\bar{Y})^2\Big[{w_1}^2\left(\frac{1}{\theta'} + {c_y}^2 + 3\lambda_a^2 C_x^2 - 4\lambda_a C_{yx}\right) + {w_2}^2\left(\frac{1}{\theta'} + {c_y}^2 + {g_1}^2\lambda_a^2 C_x^2 + 4g_1\lambda_a C_{yx}\right) + 2w_1w_2\Big(\frac{1}{\theta'} + {c_y}^2 +$
$(1-g_1)\lambda_a^2 C_x^2 - 2\lambda_a C_{yx} + 2g_1\lambda_a C_{yx}\Big) - 2w_1\left(\frac{1}{\theta'} + \lambda_a^2 C_x^2 - \lambda_a C_{yx}\right) - 2w_2\left(\frac{1}{\theta'} + g_1\lambda_a C_{yx}\right) + \frac{1}{\theta'}\Big]$
(71)

$\boldsymbol{T^*}_{\boldsymbol{DS11}} = \theta'(\bar{Y})^2\Big[{w_1}^2\left(\frac{1}{\theta'} + {c_y}^2 + \lambda_b^2 C_z^2\right) + {w_2}^2\left(\frac{1}{\theta'} + {c_y}^2 + 3{g_2}^2\lambda_a^2 C_z^2\right) + 2w_1w_2\left(\frac{1}{\theta'} + {c_y}^2 + {g_2}^2\lambda_b^2 C_z^2\right) - 2w_1\left(\frac{1}{\theta'}\right) -$
$2w_2\left(\frac{1}{\theta'} + {g_2}^2\lambda_b^2 C_z^2\right) + \frac{1}{\theta'}\Big]$ (72)

$\boldsymbol{T^*}_{\boldsymbol{DS12}} = \theta'(\bar{Y})^2\Big[{w_1}^2\left(\frac{1}{\theta'} + {c_y}^2 + \alpha_1(2\alpha_1+1)\lambda_a^2 c_x^2 - 4\,\alpha_1\lambda_a C_{yx}\right) + {w_2}^2\left(\frac{1}{\theta'} + {c_y}^2 + \delta_1(2\delta_1-1){g_1}^2\lambda_a^2 c_x^2 + 4g_1\delta_1\lambda_1 C_{yx}\right) +$
$2w_1w_2\Big(\frac{1}{\theta} + {c_y}^2 + \frac{\delta_1(\delta_1-1)}{2}{g_1}^2\lambda_a^2 c_x^2 + \frac{\alpha_1(\alpha_1+1)}{2} - \alpha_1 g_1\delta_1)\lambda_a^2 c_x^2 - 2\alpha_1\lambda_1 C_{yx} + +2g_1\delta_1\lambda_a C_{yx}\Big) - 2w_1\Big(\frac{1}{\theta} + \frac{\alpha_1(\alpha_1+1)}{2}\lambda_a^2 c_x^2 -$
$\alpha_1\lambda_a C_{yx}\Big) - 2w_2\left(\frac{1}{\theta} + \frac{\delta_1(\delta_1-1)}{2}{g_2}^2\lambda_a^2 c_x^2 + g_1\delta_1\lambda_a C_{yx}\right) + \frac{1}{\theta'}\Big]$ (73)

$\boldsymbol{T^*}_{\boldsymbol{DS13}} = \theta'(\bar{Y})^2\Big[{w_1}^2\left(\frac{1}{\theta'} + {c_y}^2 + \alpha_2(2\alpha_2-1)\lambda_b^2 c_z^2\right) + {w_2}^2\left(\frac{1}{\theta'} + {c_y}^2 + \delta_2(2\delta_2+1){g_2}^2\lambda_b^2 c_z^2\right) + 2w_1w_2\Big(\frac{1}{\theta'} + {c_y}^2 +$
$\left(\frac{\alpha_2(\alpha_2-1)}{2} + \alpha_2 g_2\delta_2\right)\lambda_b^2 c_z^2 - \frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_b^2 c_z^2\Big) - 2w_1\left(\frac{1}{\theta'} + \frac{\alpha_2(\alpha_2-1)}{2}\lambda_b^2 c_z^2\right) - 2w_2\left(\frac{1}{\theta'} - \frac{\delta_2(\delta_2+1)}{2}{g_2}^2\lambda_b^2 c_z^2\right) + \frac{1}{\theta'}\Big]$
(74)

$\boldsymbol{T^*}_{\boldsymbol{DS14}} = \theta'(\bar{Y})^2\Big[{w_1}^2\left(\frac{1}{\theta'} + {c_y}^2 + 3\lambda_1^2 C_x^2 + \lambda_2^2 c_z^2 - 4\lambda_1 C_{yx}\right) + {w_2}^2\left(\frac{1}{\theta'} + {c_y}^2 + {g_1}^2\lambda_1^2 C_x^2 + 3{g_2}^2\lambda_2^2 c_z^2 + 4g_1\lambda_1 C_{yx}\right) +$
$2w_1w_2\Big(\frac{1}{\theta'} + {c_y}^2 - {g_1}^2\lambda_1^2 c_x^2 + {g_2}^2\lambda_2^2 c_z^2 + (1-g_1)\lambda_1^2 C_x^2 - 2\lambda_1 C_{yx} + 2\lambda_2 C_{yz} + 2g_1\lambda_1 C_{yx}\Big) - 2w_1\left(\frac{1}{\theta'} + \lambda_1^2 C_x^2 - \lambda_1 C_{yx}\right) -$
$2w_2\left(\frac{1}{\theta'} - {g_2}^2\lambda_2^2 c_z^2 + g_1\lambda_1 C_{yx}\right) + \frac{1}{\theta'}\Big]$ (75)

$\boldsymbol{T^*}_{\boldsymbol{DS15}} = \theta'(\bar{Y})^2\Big[{w_1}^2\left(\frac{1}{\theta'} + {c_y}^2 + 3\lambda_a^2 C_x^2 + \lambda_b^2 c_z^2 - 4\lambda_1 C_{yx}\right) + {w_2}^2\left(\frac{1}{\theta'} + {c_y}^2 + {g_1}^2\lambda_a^2 C_x^2 + 3{g_2}^2\lambda_b^2 c_z^2 + 4g_1\lambda_a C_{yx}\right) +$
$2w_1w_2\Big(\frac{1}{\theta'} + {c_y}^2 - {g_1}^2\lambda_a^2 c_x^2 + {g_2}^2\lambda_b^2 c_z^2 + (1-g_1)\lambda_a^2 C_x^2 + (g_1-g_2)\lambda_a\lambda_b C_{yz} + -2\lambda_a C_{yx} + 2g_1\lambda_a C_{yx}\Big) - 2w_1\Big(\frac{1}{\theta'} + \lambda_a^2 C_x^2 -$
$\lambda_a C_{yx}\Big) - 2w_2\left(\frac{1}{\theta'} - {g_2}^2\lambda_b^2 c_z^2 + g_1\lambda_a C_{yx}\right) + \frac{1}{\theta'}\Big]$ (76)

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$\boldsymbol{T^*_{DS16}}$ The MSE of T^*_{DS16} is that of the case II mother estimator and it is as presented in equation (53)

4.2 Evaluation of efficiency

Case I:

Let $MSE(T_{DS1}), MSE(T_{DSj})_{opt}$ be the Mean Square Errors sample mean estimator $\bar{y}$ and that of the suggested efficient family of ratio-cum-product estimators under two phase sampling scheme for case I, then $MSE(T_{DSj})_{opt}$ shall be deemed to be more efficient than$MSE(T_{DS1})$or any other member of the suggested estimator, if the following conditions holds:

$$\left(\frac{MSE(T_{DSj})opt}{MSE(T_{DS1})}\right)^{-1} > 1, \text{ or if} \left(\frac{(\bar{Y})^2\left[1+\frac{(2Q_3Q_4Q_5-Q_2Q_4^2-Q_1Q_5^2)}{(Q_1Q_2-Q_3^2)}\right]}{[\theta(\bar{Y})^2{c_y}^2]}\right) < 1 \qquad (77)$$

Case II:

Simlilarly Let $MSE(T^*_{DS1})$, $MSE(T^*_{DS})_{opt}$ be the Mean Square Errors of the sample mean estimator $\bar{y}$ and that of the suggested efficient family of ratio-cum-product estimators under two phase sampling scheme for case II respectively, then;$MSE(T^*_{DS})_{opt}$ shall be said to be more efficient than $MSE(T^*_{DS1})$, if;

$$\left(\frac{MSE(T^*_{DSj})opt}{MSE(T^*_{DS1})}\right)^{-1} > 1, \text{ or if} \left(\frac{(\bar{Y})^2\left[1+\frac{(2U_3U_4U_5-U_2U_4^2-U_1U_5^2)}{(U_1U_2-U_3^2)}\right]}{[\theta'(\bar{Y})^2{c_y}^2]}\right) < 1 \qquad (78)$$

The $MSE(T^*_{DS})_{opt}$under case II, shall be deemed to be Percentage Relative Efficient to $MSE(T_{DSj})_{opt}$ case I or any other member of the suggested estimator, if the following conditions holds:

$$\left(\frac{(\bar{Y})^2\left[1+\frac{(2Q_3Q_4Q_5-Q_2Q_4^2-Q_1Q_5^2)}{(Q_1Q_2-Q_3^2)}\right]}{[\theta(\bar{Y})^2{c_y}^2]}\right) \times 100 - \left(\frac{(\bar{Y})^2\left[1+\frac{(2U_3U_4U_5-U_2U_4^2-U_1U_5^2)}{(U_1U_2-U_3^2)}\right]}{[\theta'(\bar{Y})^2{c_y}^2]}\right) \times 100 > 0 \qquad (79)$$

4.3 Empirical study

In order to verify the veracity of the performances of the suggested estimator and the accuracy of the theoretical proposition of the study, Four (4) Agricultural data sets were employed from secondary sources. The Agricultural datasets alongside their sources is as presented below:

Population I. *Source: Sukhatme and Chand (1977)*
Y: Apple trees of bearing age in 1964
X: Bushels of apples harvested in 1964
Z: Bushels of apples harvested in 1959

$M = 200, m_1 = 20, m_2 = 30$, $\bar{Y} = 1031.82$, $\bar{X} = 2934.58, \bar{Z} = 3651.49,$
$\rho_{yx} = 0.93$, $\rho_{yz} = 0.77, \rho_{xz} = 0.84, {c_y}^2 = 2.55280, {c_x}^2 = 4.02504,$, ${c_z}^2 = 2.09379$

Population II. *Source: Murthy (1967)*
Y: Area under wheat in 1964
X: Area under wheat in 1963
Z: cultivated area in1961

$M = 34, m_1 = 7, m_2 = 10$, $\bar{Y} = 199.44$ acre, $\bar{X} = 208.89$ acre, $\bar{Z} = 747.59$ acre,
$\rho_{yx} = 0.9801, \rho_{yz} = 0.9043, \rho_{xz} = 0.9097, {c_y}^2 = 0.5673, {c_x}^2 = 0.5191, {c_z}^2 = 0.3527$

Population III. *Source: Khare and Rehman (2015)*
Y: Number of Agricultural labour
X: Area of village hectares

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Z: Number of cultivators in the village

$M = 96, m_1 = 24, m_2 = 30, \bar{Y} = 137.9271, \bar{X} = 144.8720, \bar{Z} = 185.188,$
$\rho_{yx} = 0.786, \rho_{yz} = 0.786, , \rho_{xz} = 0.819, c_y{}^2 = 1.7509, c_x{}^2 = 0.6585, , c_z{}^2 = 2.4090$

Population IV. *Source: Steel and Torrie (1960)*
Y: Log of leaf burn in seconds
X: Potassium percentage
Z: Chlorine percentage

$M = 30, m_1 = 6, m_2 = 9, \bar{Y} = 0.6860, \bar{X} = 4.6437, \bar{Z} = 0.8077,$
$\rho_{yx} = 0.1794, \rho_{yz} = -0.4996, , \rho_{xz} = 0.4074, c_y{}^2 = 0.4803, c_x{}^2 = 0.2295, c_z{}^2 = 0.7493$

Table 4: Results of the biases and relative biases of the suggested estimator T_{DSj} case I

		Populations			
Fixed Scalars	T_{DSj}	**I**	**II**	**III**	**IV**
$a = b = \alpha_1 = \alpha_2 = \delta_1 = \delta_2 = 0$	T_{DS1}	0.00(0.00)	0.00(0.00)	0.00(0.00)	0.00(0.00)
$a = b = \alpha_1 = \alpha_2 = 1, \delta_1 = \delta_2 = 0$	T_{DS2}	17.8579(0.01730)	0.05638(0.000282)	3.6438(0.02642)	0.02750(0.04009)
$a = b = 1, \alpha_1 = \alpha_2 = 0, \delta_1 = \delta_2 = 1$	T_{DS3}	0.4105(0.000397)	0.03105(0.000155)	0.8496(0.00616)	0.2374(0.34606)
$a = b = 1, \alpha_1 = \alpha_2 = 0, \delta_1 = \delta_2 = 1$	T_{DS4}	0.6037(0.000585)	0.3427(0.0017183)	-0.1555(-0.00113)	-0.13015(-0.1897)
$a = b = 1, \alpha_1 = \alpha_2 = \delta_1 = \delta_2 = 0$	T_{DS5}	-231.108(-0.2239)	-9.2948(-0.04660)	-12.1458(-0.0880)	-0.02598(-0.0378)
$a = b = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = \delta_1 = -1$	T_{DS6}	-744.449(-0.7215)	-58.5739(-0.2937)	-59.8053(-0.4336)	-0.12204(-0.1779)
$a = b = \alpha_2 = \delta_2 = 1, \alpha_1 = \delta_1 = 0$	T_{DS7}	-668.574(-0.6479)	-63.1002(-0.6479)	-86.4832(-0.6271)	-0.18999(-0.2768)
$a = b = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = \delta_1 = -1$	T_{DS8}	-744.449(-0.7215)	-58.5739(-0.2937)	-59.8053(-0.4336)	-0.12204(-0.1779)
$a = b = 1, \alpha_2 = \delta_2 = -1, \alpha_1 = \delta_1 = 0$	T_{DS9}	-673.291(-0.6525)	-60.1600(-0.3016)	-82.9530(-0.6015)	-0.1770(-0.2580)
$a = b = \delta_1 = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = -1$	T_{DS10}	-786.013(-0.7617)	-82.1420(-0.4118)	-69.9057(-0.5068)	-0.6778(-0.9880)
$a = b = \alpha_2 = \delta_2 = 1, \alpha_1 = \delta_1 = 0$	T_{DS11}	-668.574(-0.6479)	-63.1002(-0.6479)	-86.4832(-0.6271)	-0.18999(-0.2768)
$a = b = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = -0.5, \delta_1 = -1$	T_{DS12}	-595.602(-0.5772)	-28.6863(-0.1438)	-37.6629(-0.2730)	-0.03095(-0.0451)
$a = b = 1, \alpha_2 = 0.5, \delta_2 = -1, \alpha_1 = \delta_1 = 0$	T_{DS13}	-541.404(-0.5247)	-39.2408(-0.1967)	-63.4422(-0.4599)	-0.1131(-0.1646)
$a = b = 1, \alpha_1 = \delta_2 = -1, \alpha_2 = \delta_1 = 1$	T_{DS14}	-855.079(-0.8287)	-105.511(-0.5290)	-97.4809(-0.7067)	-0.2242(-0.3268)
$a = b = 1, \alpha_1 = \delta_2 = 0.5, \alpha_2 = \delta_1 = -1$	T_{DS15}	-850.585(-0.8244)	-89.1347(-0.4469)	-84.7628(-0.6145)	-0.1463(-0.2132)
$a = b = \delta_2 = 1, \delta_1 = -0.5, \alpha_1 = -1, \alpha_2 = 0.5$	T_{DS16}	-818.515(-0.7933)	-91.1115(-0.4568)	-93.1007(-0.6750)	-0.1844(-0.2688)

Table 5: Results of the biases and relative biases of the suggested estimator T_{DSj} case II

		Populations			
Fixed Scalars	$T^*{}_{DSj}$	**I**	**II**	**III**	**IV**
$a = b = \alpha_1 = \alpha_2 = \delta_1 = \delta_2 = 0$	$T^*{}_{DS1}$	0.00(0.00)	0.00(0.00)	0.00(0.00)	0.00(0.00)
$a = b = \alpha_1 = \alpha_2 = 1, \delta_1 = \delta_2 = 0$	$T^*{}_{DS2}$	30.4318(0.02949)	-0.1796(-0.000900)	-0.79994(-0.00580)	0.01068(0.015568)
$a = b = 1, \alpha_1 = \alpha_2 = 0, \delta_1 = \delta_2 = 1$	$T^*{}_{DS3}$	0.4795(0.000464)	0.1229(0.0006162)	-0.2415(0.001751)	-0.09874(-0.14393)
$a = b = 1, \alpha_1 = \alpha_2 = 0, \delta_1 = \delta_2 = 1$	$T^*{}_{DS4}$	0.5447(0.00052)	0.1842(0.0009235)	0.02611(0.000189)	0.00034(0.000495)
$a = b = 1, \alpha_1 = \alpha_2 = \delta_1 = \delta_2 = 0$	$T^*{}_{DS5}$	-158.675(-0.153)	-5.8871(-0.029518)	-5.9865(-0.434055)	-0.01808(-0.02635)
$a = b = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = \delta_1 = -1$	$T^*{}_{DS6}$	-746.390(-0.723)	-57.6475(-0.28904)	-58.5628(-0.42461)	-0.04904(-0.07148)
$a = b = \alpha_2 = \delta_2 = 1, \alpha_1 = \delta_1 = 0$	$T^*{}_{DS7}$	-80.8196(-0.078)	-8.0079(-0.040151)	-8.7268(-0.063274)	-0.03878(-0.05653)
$a = b = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = \delta_1 = -1$	$T^*{}_{DS8}$	-746.3903(-0.7233)	-57.6475(-0.28904)	-58.5628(-0.42461)	-0.04904(-0.07148)
$a = b = 1, \alpha_2 = \delta_2 = -1, \alpha_1 = \delta_1 = 0$	$T^*{}_{DS9}$	-156.4912(-0.1516)	-6.4983(-0.032582)	-6.6624(-0.048306)	-0.02511(-0.0366)
$a = b = \delta_1 = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = -1$	$T^*{}_{DS10}$	-788.5351(-0.7642)	-81.4031(-0.40815)	-69.0183(-0.50042)	-0.06830(-0.0995)
$a = b = \alpha_2 = \delta_2 = 1, \alpha_1 = \delta_1 = 0$	$T^*{}_{DS11}$	-80.8196(-0.07833)	-8.00797(-0.04015)	-8.7268(-0.063274)	-0.0387(-0.05641)
$a = b = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = -0.5, \delta_1 = -1$	$T^*{}_{DS12}$	-579.742(-0.56186)	-25.7227(-0.12897)	-34.0190(-0.24665)	0.03095(-0.04511)
$a = b = 1, \alpha_2 = 0.5, \delta_2 = -1, \alpha_1 = \delta_1 = 0$	$T^*{}_{DS13}$	-146.545(-0.14203)	-6.2837(-0.031506)	-6.5255(-0.047316)	-0.02378(-0.0346)
$a = b = 1, \alpha_1 = \delta_2 = -1, \alpha_2 = \delta_1 = 1$	$T^*{}_{DS14}$	-788.2009(-0.7638)	-81.7233(-0.40976)	-69.4523(-0.50356)	-0.02356(-0.0343)
$a = b = 1, \alpha_1 = \delta_2 = 0.5, \alpha_2 = \delta_1 = -1$	$T^*{}_{DS15}$	-780.127(-0.75068)	-76.716(-0.384657)	-67.1961(-0.48721)	-0.06949(-0.1012)
$a = b = \delta_2 = 1, \delta_1 = -0.5, \alpha_1 = -1, \alpha_2 = 0.5$	$T^*{}_{DS16}$	-753.815(-0.73056)	-65.4023(-0.32792)	-62.4124(-0.45252)	-0.0618(-0.09008)

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Remark 1: The relative biases are the values that are enclosed in braces. The estimator T_{DS1} with zero bias and relative bias is the sample mean which is an unbiased estimator of the population mean. The estimators T_{DS2} ,T_{DS3} andT_{DS4} with positive biases and relative biases are the ones that overestimate the population. The estimators' T_{DS5} to T_{DS16} with negative biases and relative biases are those which are less biased towards the true value of the estimate within the population.

Table 6: Results of the Mean Square Errors (MSEs) of the suggested estimator T_{DSj} case I

		Populations			
Fixed Scalars	T_{DSj}	**I**	**II**	**III**	**IV**
$a = b = \alpha_1 = \alpha_2 = \delta_1 = \delta_2 = 0$	T_{DS1}	122303.02	2559.91	2215.26	0.030137
$a = b = \alpha_1 = \alpha_2 = 1, \delta_1 = \delta_2 = 0$	T_{DS2}	1224773.64	19299.36	17468.39	0.238268
$a = b = 1, \alpha_1 = \alpha_2 = 0, \delta_1 = \delta_2 = 1$	T_{DS3}	80409.38	811.44	1052.93	0.022319
$a = b = 1, \alpha_1 = \alpha_2 = 0, \delta_1 = \delta_2 = 1$	T_{DS4}	80409.38	811.44	1052.93	0.022319
$a = b = 1, \alpha_1 = \alpha_2 = \delta_1 = \delta_2 = 0$	T_{DS5}	238461.94	1853.76	1675.23	0.017824
$a = b = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = \delta_1 = -1$	T_{DS6}	767717.18	11732.55	8263.94	0.090620
$a = b = \alpha_2 = \delta_2 = 1, \alpha_1 = \delta_1 = 0$	T_{DS7}	692950.66	12802.15	12125.23	0.136261
$a = b = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = \delta_1 = -1$	T_{DS8}	767717.18	11732.55	8263.94	0.096156
$a = b = 1, \alpha_2 = \delta_2 = -1, \alpha_1 = \delta_1 = 0$	T_{DS9}	694715.56	11998.47	11441.47	0.121441
$a = b = \delta_1 = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = -1$	T_{DS10}	809192.61	16211.09	9602.18	0.053090
$a = b = \alpha_2 = \delta_2 = 1, \alpha_1 = \delta_1 = 0$	T_{DS11}	692950.67	12802.15	12125.23	0.136261
$a = b = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = -0.5, \delta_1 = -1$	T_{DS12}	615924.04	5810.85	5190.86	0.020146
$a = b = 1, \alpha_2 = 0.5, \delta_2 = -1, \alpha_1 = \delta_1 = 0$	T_{DS13}	558630.93	7826.18	8750.39	0.077600
$a = b = 1, \alpha_1 = \delta_2 = -1, \alpha_2 = \delta_1 = 1$	T_{DS14}	880986.74	20909.07	13422.54	0.153515
$a = b = 1, \alpha_1 = \delta_2 = 0.5, \alpha_2 = \delta_1 = -1$	T_{DS15}	846150.22	17712.52	11678.55	0.100352
$a = b = \delta_2 = 1, \delta_1 = -0.5, \alpha_1 = -1, \alpha_2 = 0.5$	T_{DS16}	846514.13	18383.53	13024.49	0.132538

Table 7: Results of the Mean Square Errors (MSEs) of the suggested estimator T_{DSj} case II

		Populations			
Fixed Scalars	$T^*{}_{DSj}$	**I**	**II**	**III**	**IV**
$a = b = \alpha_1 = \alpha_2 = \delta_1 = \delta_2 = 0$	$T^*{}_{DS1}$	77005.61	1592.83	1040.90	0.020719
$a = b = \alpha_1 = \alpha_2 = 1, \delta_1 = \delta_2 = 0$	$T^*{}_{DS2}$	810763.64	11922.84	7515.22	0.152528
$a = b = 1, \alpha_1 = \alpha_2 = 0, \delta_1 = \delta_2 = 1$	$T^*{}_{DS3}$	69345.49	1402.09	1123.84	0.024507
$a = b = 1, \alpha_1 = \alpha_2 = 0, \delta_1 = \delta_2 = 1$	$T^*{}_{DS4}$	69345.49]	1402.09	1123.84	0.024507
$a = b = 1, \alpha_1 = \alpha_2 = \delta_1 = \delta_2 = 0$	$T^*{}_{DS5}$	163724.63	1174.12	825.70	0.012402
$a = b = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = \delta_1 = -1$	$T^*{}_{DS6}$	769873.32	11531.54	8084.89	0.033468
$a = b = \alpha_2 = \delta_2 = 1, \alpha_1 = \delta_1 = 0$	$T^*{}_{DS7}$	96289.97	1774.16	1423.54	0.031801
$a = b = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = \delta_1 = -1$	$T^*{}_{DS8}$	769873.32	11531.54	8084.89	0.033468
$a = b = 1, \alpha_2 = \delta_2 = -1, \alpha_1 = \delta_1 = 0$	$T^*{}_{DS9}$	161470.62	1296.02	918.94	0.017226
$a = b = \delta_1 = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = -1$	$T^*{}_{DS10}$	812472.75	16124.91	9500.21	0.046600
$a = b = \alpha_2 = \delta_2 = 1, \alpha_1 = \delta_1 = 0$	$T^*{}_{DS11}$	96289.97	1774.16	1423.54	0.031801
$a = b = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = -0.5, \delta_1 = -1$	$T^*{}_{DS12}$	601081.47	5260.38	4720.78	0.015457
$a = b = 1, \alpha_2 = 0.5, \delta_2 = -1, \alpha_1 = \delta_1 = 0$	$T^*{}_{DS13}$	151208.25	1253.24	900.05	0.016167
$a = b = 1, \alpha_1 = \delta_2 = -1, \alpha_2 = \delta_1 = 1$	$T^*{}_{DS14}$	812131.33	16190.09	9560.55	0.059109
$a = b = 1, \alpha_1 = \delta_2 = 0.5, \alpha_2 = \delta_1 = -1$	$T^*{}_{DS15}$	804533.24	15253.97	9259.92	0.047647
$a = b = \delta_2 = 1, \delta_1 = -0.5, \alpha_1 = -1, \alpha_2 = 0.5$	$T^*{}_{DS16}$	775187.36	12994.97	8493.24	0.037151

Remark 2: Given a class or family of estimators, the estimators with smaller MSEs are deemed to be efficient. Thus Estimators $T_{DS1}T_{DS3}$,T_{DS4} ,T_{DS5} andT_{DS12} are efficient in case I and estimators $T^*{}_{DS1}T^*{}_{DS3}$,$T^*{}_{DS4}$,$T^*{}_{DS4}$ $T^*{}_{DS9}$,$T^*{}_{DS12}$and$T^*{}_{DS13}$ are efficient in case II.

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Table 8: Results of the Percent Relative Efficiencies (PREs) of the suggested estimator T_{DSj} case I

		Populations			
Fixed Scalars	T_{DSj}	**I**	**II**	**III**	**IV**
$a = b = \alpha_1 = \alpha_2 = \delta_1 = \delta_2 = 0$	T_{DS1}	100	100	100	100
$a = b = \alpha_1 = \alpha_2 = 1, \delta_1 = \delta_2 = 0$	T_{DS2}	1001.43	753.91	788.55	790.62
$a = b = 1, \alpha_1 = \alpha_2 = 0, \delta_1 = \delta_2 = 1$	T_{DS3}	65.75	31.70	47.53	74.06
$a = b = 1, \alpha_1 = \alpha_2 = 0, \delta_1 = \delta_2 = 1$	T_{DS4}	65.75	31.70	47.53	74.06
$a = b = 1, \alpha_1 = \alpha_2 = \delta_1 = \delta_2 = 0$	T_{DS5}	194.98	72.42	75.62	59.14
$a = b = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = \delta_1 = -1$	T_{DS6}	627.72	458.32	373.04	300.70
$a = b = \alpha_2 = \delta_2 = 1, \alpha_1 = \delta_1 = 0$	T_{DS7}	566.59	500.10	547.35	452.14
$a = b = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = \delta_1 = -1$	T_{DS8}	627.72	458.32	373.04	319.06
$a = b = 1, \alpha_2 = \delta_2 = -1, \alpha_1 = \delta_1 = 0$	T_{DS9}	568.03	468.71	516.48	402.96
$a = b = \delta_1 = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = -1$	T_{DS10}	661.63	633.27	433.46	176.16
$a = b = \alpha_2 = \delta_2 = 1, \alpha_1 = \delta_1 = 0$	T_{DS11}	566.59	500.10	547.35	452.14
$a = b = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = -0.5, \delta_1 = -1$	T_{DS12}	503.61	226.99	234.32	66.85
$a = b = 1, \alpha_2 = 0.5, \delta_2 = -1, \alpha_1 = \delta_1 = 0$	T_{DS13}	456.76	305.72	395.01	257.49
$a = b = 1, \alpha_1 = \delta_2 = -1, \alpha_2 = \delta_1 = 1$	T_{DS14}	720.33	816.79	605.91	509.39
$a = b = 1, \alpha_1 = \delta_2 = 0.5, \alpha_2 = \delta_1 = -1$	T_{DS15}	691.85	691.92	527.19	332.99
$a = b = \delta_2 = 1, \delta_1 = -0.5, \alpha_1 = -1, \alpha_2 = 0.5$	T_{DS16}	692.14	718.13	587.95	439.79

Table 9: Results of the Percent Relative Efficiencies (PREs) of the suggested estimator T_{DSj} case II

		Populations			
Fixed Scalars	T^*_{DSj}	**I**	**II**	**III**	**IV**
$a = b = \alpha_1 = \alpha_2 = \delta_1 = \delta_2 = 0$	T^*_{DS1}	100	100	100	100
$a = b = \alpha_1 = \alpha_2 = 1, \delta_1 = \delta_2 = 0$	T^*_{DS2}	1052.86	748.53	721.99	736.17
$a = b = 1, \alpha_1 = \alpha_2 = 0, \delta_1 = \delta_2 = 1$	T^*_{DS3}	90.05	88.03	107.97	118.28
$a = b = 1, \alpha_1 = \alpha_2 = 0, \delta_1 = \delta_2 = 1$	T^*_{DS4}	90.05	88.03	107.97	118.28
$a = b = 1, \alpha_1 = \alpha_2 = \delta_1 = \delta_2 = 0$	T^*_{DS5}	212.61	73.71	79.33	59.85
$a = b = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = \delta_1 = -1$	T^*_{DS6}	999.76	723.97	776.72	161.53
$a = b = \alpha_2 = \delta_2 = 1, \alpha_1 = \delta_1 = 0$	T^*_{DS7}	125.04	111.38	136.76	153.49
$a = b = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = \delta_1 = -1$	T^*_{DS8}	999.76	723.97	776.72	161.53
$a = b = 1, \alpha_2 = \delta_2 = -1, \alpha_1 = \delta_1 = 0$	T^*_{DS9}	209.69	81.37	88.28	83.14
$a = b = \delta_1 = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = -1$	T^*_{DS10}	1055.08	1012.34	912.69	224.91
$a = b = \alpha_2 = \delta_2 = 1, \alpha_1 = \delta_1 = 0$	T^*_{DS11}	125.04	111.38	136.76	153.49
$a = b = 1, \alpha_2 = \delta_2 = 0, \alpha_1 = -0.5, \delta_1 = -1$	T^*_{DS12}	780.57	330.25	453.53	74.60
$a = b = 1, \alpha_2 = 0.5, \delta_2 = -1, \alpha_1 = \delta_1 = 0$	T^*_{DS13}	196.36	78.68	86.47	78.02
$a = b = 1, \alpha_1 = \delta_2 = -1, \alpha_2 = \delta_1 = 1$	T^*_{DS14}	1054.63	1016.44	918.49	285.28
$a = b = 1, \alpha_1 = \delta_2 = 0.5, \alpha_2 = \delta_1 = -1$	T^*_{DS15}	1044.77	957.66	889.61	229.97
$a = b = \delta_2 = 1, \delta_1 = -0.5, \alpha_1 = -1, \alpha_2 = 0.5$	T^*_{DS16}	1006.66	815.84	815.95	179.31

Remark 2: Given a class or family of estimators, the estimators that are percentage efficient are ones with smaller PREs in relation to the sample mean estimator T_{DS1}. Thus; estimators T_{DS3}, T_{DS4}, T_{DS5} and T_{DS12} are percentage efficient in relation to the sample mean estimator T_{DS1} in case I and estimators T^*_{DS3}, T^*_{DS4}, T^*_{DS5}, T^*_{DS9}, T^*_{DS12} and T^*_{DS13} are percentage efficient in relation to the sample mean T^*_{DS1} in case II. The estimators in case II are more efficient than those of case I. Implying that sub-sampling the second sample independent of the first sample is advantageous and give rise to gain in efficiency and superiority over the first phase in double sampling.

4.4 Discussion of result

Efficient classes of ratio –cum-product estimators of population mean in Two Phase Sampling in presence of two auxiliary variables was proposed and presented in (10). The (10) was then transformed to an expanded form and presented in (14), from where various attributes such bias, relative bias, MSE and optimal MSE of the estimator were derived and shown in (18), (19), (21) and (35) respectively for the case I and in (51), (52), (53) and (60) each for the case II of the double sampling strategy.

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The members of the proposed class of estimator T_{DSj} of population mean $\bar{Y}$ of the study variable $\bar{y}$ were obtained by varying the values of the scalars that helps in proposing the estimator T_{DS_j}. The scalars were uniquely varied in a manner that more brethren of ratio-cum-product estimators were obtained and presented in table 1, from where it was observed that the estimator produces the traditional sample mean ratio estimator $\bar{y}$, the dual to Singh and Tailor (2005) estimator due to Tailor et al (2012), Singh and Tailor (2011) generalized version of the dual to ratio-cum-product estimator of the population mean $\bar{Y}$ were among the members of the proposed estimator. The biases and relative biases of all the members of the estimator case I were then derived and presented in table 2. Also, the MSEs of all the members of the estimators case I, were derived, presented and labeled as (36),(37),(38),…(50) each.

In like manner, the attributes such as biases of members of the estimators for case II, were derive and showcased in table 3, while the MSEs of all the members of the estimators case II, were equally obtained and can be envisage in equations (62),(63),(64),…(76) respectively. Conditions for evaluating the efficiency of an estimator in relation to the conventional sample mean estimator was established and presented in (77), (78) for case I and case II respectively. In order to verify the veracity of the performances of the suggested estimator and the accuracy of the theoretical proposition of the study, Four (4) Agricultural data sets were employed from secondary sources.

A close look at the outcome of the empirical study revealed that the estimator T_{DS1} with zero bias and relative bias is the sample mean which is an unbiased estimator of the population mean. The estimators T_{DS2} , T_{DS3} and T_{DS4} with positive biases and relative biases are the ones that overestimate the population. The estimators' T_{DS5} to T_{DS16} with negative biases and relative biases are those which are less biased towards the true value of the estimate within the population for cases I and II. This inferences were drawn from table4 and table 5 each. Furthermore, the estimators $T_{DS1} T_{DS3}$, T_{DS4} , T_{DS5} and T_{DS12}, were found to be more efficient in case I and estimators' $T^*_{DS1} T^*_{DS3}, T^*_{DS4}, T^*_{DS4}$ T^*_{DS9}, T^*_{DS12} and T^*_{DS13} were more efficient in case II, having been ascertained to have produced smaller MSEs and can be seen on table 6 and table 7 for cases I and II of the results.

To gain more inside into the performances of the estimators, the PREs of the estimators were calculated and the result showed that the estimators T_{DS3} , T_{DS4} , T_{DS5} and T_{DS12} in case I, as shown in table 8,are percentage efficient in relation to the sample mean estimator T_{DS1} since they are the ones with smaller PREs in case I. Similarly, the estimators T^*_{DS3}, T^*_{DS4}, T^*_{DS5} , T^*_{DS9}, T^*_{DS12} and T^*_{DS13} as presented in table 9 are percentage efficient in relation to the sample mean T^*_{DS1} in case II. The estimators in case II are more efficient than those of case I. Implying that sub-sampling the second sample independent of the first sample is advantageous and give rise to gain in efficiency and superiority over the first phase in double sampling scheme.

5. Conclusion

In this study, an efficient classes of ratio –cum-product estimators of population mean in Two Phase Sampling in presence of two auxiliary variables for cases I and II was proposed. The biases, relative biases, MSEs and the optimal conditions for both cases were obtained. Empirical study was conducted using Four (4) Agricultural data sets in order to ascertain the veracity of the established theoretical proposition, from where it was found from the results that various members of the estimators showed appreciable gain in efficiency. The efficiency increases as the sample size reduces. The gain in efficiency was more significant in case II. Therefore, sub-sampling independent of the first phase sample is recommended for a higher efficiency.

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