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Two Solutions for the Work-Kinetic Energy Theorem and the Principle of Conservation of Energy

[1]Benjamín Vázquez González, [2]Jose Luis Ramírez Cruz, [3]Homero Jiménez Rabiela, [4]María Berenice Guadalupe Quintana Díaz

[1,2,3,4]Universidad Autónoma Metropolitana, Unidad Azcapotzalco, Av. San Pablo Xalpa 420, Col. Reynosa Tamaulipas, Azcapotzalco, México

***Abstract* - This paper presents a complete analytical development of one of the most frequently used problems, in describing the way in which the energy method is applied on the solution of problems on basic physics and engineering courses. The majority of the classical references do not explain the meaning of finding two solutions when using the principle of conservation of mechanical energy. The classical references just consider the existence of one real solution. The problem corresponds to the displacement of a body on an inclined plane with dry friction, the body moves by the action of gravity and by an initial condition of velocity that is given to it. At the lower end of the inclined plane there is a previously compressed spring, that produces an initial force when the body comes into contact with the spring, it is required to determine the maximum displacement that the body will reach, once it comes into contact with the spring. The use of the Newton equations completes and contributes to explain the meaning of the second solution of the problem, to which corresponds with the real behavior of the system.**

Keywords: Energy conservation, Mechanical work, Kinetic energy, Potential energy, Dissipative systems, Newton equations.

I. INTRODUCTION

The solution of problems with mechanical systems, in which the use of mechanical energy conservation is applied, requires concepts and developments of skills that allow distinguishing and attributing an energy change, for the dynamic behavior of the system.

One of the most complete problems used to illustrate the application of the principle of conservation of mechanical energy consists of a body of mass m, which is displaced by the action of gravity and with initial velocity downwards on an inclined plane, at the lower end of the inclined plane a partially compressed spring is located, there is dynamic friction between the surface of the inclined plane and the body in motion. This problem is considered in [1], it is reported only the positive part of the solution on the answers. Similar situation in [2], where there is not friction force and there is not initial velocity, in this reference, the problem is considered as an example, and the solution procedure establishes that the negative solution found is mechanical unuseful. In [3] the same problem is studied as an example, considering there is not initial velocity, and there is only one solution, without taking account that the solution comes from a second order polynomial. In reference [4], the problem is again considered as an example, and the result proposed reports a wrong solution, in this case the mistake is produced when strain energy is evaluated, and the solution just considers one root of the second order polynomial found. In all this bibliographic references the energy method is used.

In the classical bibliography, the problem of the afore mentioned mass moving over an inclined plane is posed, the strategy corresponds to the quantification of energy and work at the initial and final positions of the mass, to later calculate the variations or energy changes.

In this case, the concept of mechanical work is used to quantify external actions such as friction, it is at this moment that the definition of mechanical work is incorporated, therefore, the work-kinetic energy theorem and the principle of conservation of energy [5, 6] are used together, and then it is possible to quantify all variables since the energy perspective and determine the solution to the problem. The total energy evaluation determined from the initial and final energy variables, does not give much clarity to evaluate the changes of dynamic state along the different stages of the system's behavior.

In this way, the approach to the solution includes energy expressions, all similar between them, because they have been obtained from the work definition, but different in their conception, do not observe this, leads to obtaining imprecise assessments to quantify the energy description or nature, as mentioned above, as for example in [4].

Solving the problem by defining stages throughout the performance of the body´s motion, allows to distinguish the particular behavior in an intermediate state and therefore, to specify the way in which the elements of the system interact

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with each other, as well as, the concrete expression with which its energetic content, and its dynamic significance are evaluated. The description of the behavior of each element through the approach of the definition of work, together with the principle of conservation of energy, provides guidelines to give a complete description of the performance of each element and its interaction with the rest of the elements of the mechanical system, because the concept of work allows establishing parameters that can be interpreted from a physical perception (as the action of a force), to abstract and quantify the effect of an action. The concept of work allows approaching energy terms with direct actions described by forces, which are quantified by their effect, allowing a correlation between work and energy to be established. In accordance with [7] it is necessary to specify the difference between energy and work, this will allow defining specific changes clearly, to assess the condition of change of state of the system.

Another of the difficulties in solving problems related to the principle of conservation of mechanical energy is to verify the result. In this paper, an alternative solution is presented using the balance of forces obtained in the context of position-time, which allows validating the results, checking up the solution and also obtaining a complete interpretation of the resulting solutions. For the proposed problem, there are two real solutions (one positive and one negative, this is the main contribution of this work), however, the energy context does not facilitate the interpretation for obtaining a negative displacement, which is why the result is generally discarded, however, we demonstrate that the negative displacement solution obtained corresponds to a subsequent performance of the body of the system. It is necessary then to highlight the importance of the principle of conservation of energy, since it provides all the solutions that the problem can have, inclusive in the time-motion context.

Finally, it is also important to highlight the need to know all the aspects that the full knowledge of the principle of conservation of energy and work implies, because said principle later transcends towards more general methods such as the implementation and conceptualization of the principle of virtual work and from the Lagrange equations.

In this paper the solution of the problem is developed using two ways; the principle of conservation of mechanical energy and mechanical work, later the solution is obtained by means of Newton's equations.

In both cases the same result is obtained, however, the convergence of both solutions to the same result is not evident and this difference in the forms of the solutions allows us to explain the behavior of some of the elements that constitute the system during its dynamic performance.

II. PROBLEM STATMENT

The problem in consideration is illustrated in Figure 1.

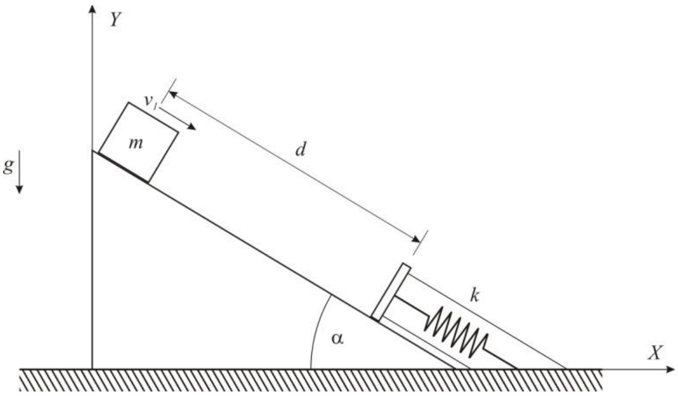

Figure 1: Inclined mass-plane system with friction and pre-compressed spring

The main problem in the quantification of the kinetic and potential energies is observed in the determination of the expressions corresponding to the changes of position in the vector fields of gravity, spring and friction, which is also due to the ambiguity in establishing a reference point. The concept of mechanical work helps to establish a quantification of the energy changes, in particular of the elastic potential energy and also by the energy dissipated by the action of friction, since the work is defined by the concept of force, which allows establishing a concrete action reference, evaluating the work allows us to define the changes in energy depending on the position in the different stages of the movement and, therefore, associate them with evaluations of a perception of time, that is, what is happening at a certain time development of the motion by the action of a force, this is a relative advantage that is found when solving the problem through the concept of mechanical work, a comparative description is made below. Establishing or defining stages in which the definition of work is used, helps to define positions to quantify energy changes.

III. SOLUTION THROUGH THE PRINCIPLE OF CONSERVATION OF MECHANICAL ENERGY

The analysis is carried out in two stages defined by three positions, this allows to find expressions of intermediate parameters that the work-energy theorem, for two extreme positions (only initial and final position), does not evaluate.

Figure 2 illustrates the first stage defined by the initial position and the intermediate position in which the body comes into contact for the first time with the support of spring k. Said spring was previously compressed to a distance δ.

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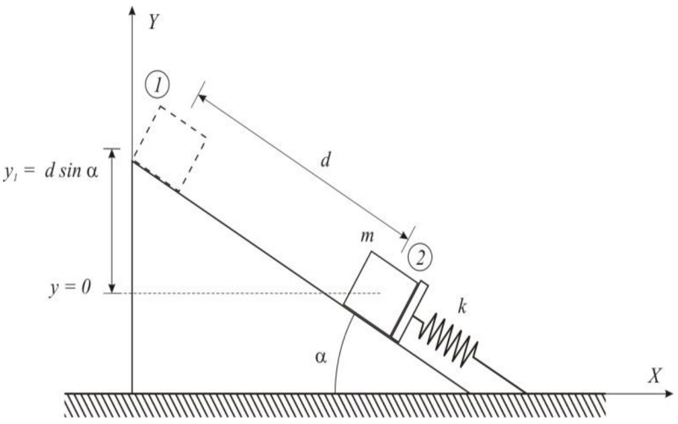

Figure 2: Pre-compressed spring-inclined plane-mass system in its first stage

It is necessary to establish a reference position to quantify the gravitational potential energy, it is chosen in position 2, that is, in position 2, y=0.

For the given configuration and to carry out the analysis of the system, the following work-kinetic energy expression is now established:

$$\Delta K_m + \Delta U_m = W_{1\to 2} \quad (1)$$

Where ΔK_m corresponds to the change in kinetic energy of the mass, ΔU_m corresponds to the change in potential energy of the mass and $W_{1\to 2}$ represents the work produced by the dissipative action of the friction force, on the stage previously defined intermediate. Then, for the system parameters, we have,

$$\frac{1}{2}mv_{f1}^2 - \frac{1}{2}mv_{i1}^2 - mgd(sin\alpha) = \int_0^d \vec{F}_f \cdot d\vec{r} \quad (2)$$

Where,

$$\int_0^d \vec{F}_f \cdot d\vec{r} = \int_0^d \mu_f \vec{N} \cdot d\vec{r} = -mgd\mu_f(cos\alpha) \quad (3)$$

From the previous equations, the expression of the velocity with which the body comes into contact with the base of the spring is determined,

$$v_{f1} = \sqrt{\frac{mv_{i1}^2 + 2mgd(sin\alpha) - 2mgd\mu_f(cos\alpha)}{m}} \quad (4)$$

Now the second stage of the motion, defined by positions 2 and 3, is analysed, Figure 3 identifies the two referred positions.

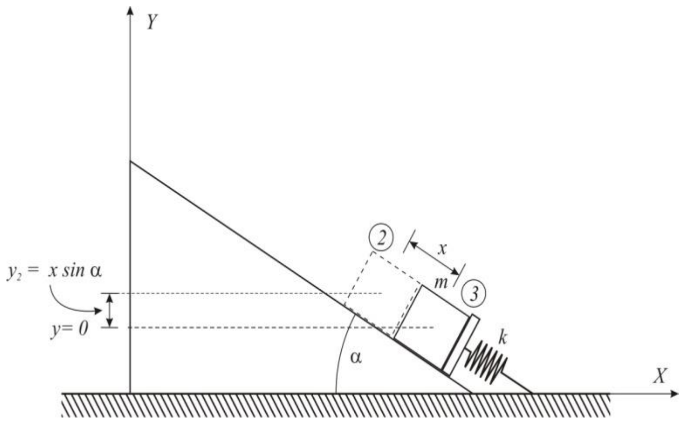

Figure 3: The mass stays in contact with the spring base

The expression for the theorem-kinetic energy is established to evaluate the present stage,

$$\Delta K_m + \Delta U_m = W_{2\to 3} + W_{ext} \quad (5)$$

Identifying the expressions that represent the energies in the respective positions and for the work-kinetic energy balance above, it results,

$$\frac{1}{2}mv_{f2}^2 - \frac{1}{2}mv_{i2}^2 - mgy_2 = \int_0^x \vec{F}_f \cdot d\vec{r} + \int_0^x \vec{F}_k \cdot d\vec{r}. \quad (6)$$

Where,

$$W_{2\to 3} = \int_0^x \vec{F}_f \cdot d\vec{r} = \int_0^x \mu_f \vec{N} \cdot d\vec{r} = \int_0^x -mg\mu_f(cos\alpha)dr = -mg\mu_f x(cos\alpha) \quad (7)$$

The expression for the external work, W_{ext}, corresponding to this stage is,

$$W_{ext} = \int_0^x \vec{F}_k \cdot d\vec{r} = \int_0^x -k(r+\delta)\,dr = -\frac{1}{2}kr^2|_0^x - k\delta r\,|_0^x = -\frac{1}{2}kx^2 - k\delta x. \quad (8)$$

Note that this expression allows us to visualize that the external work done by the spring on the body during the compression process has two terms, one quadratic in the displacement that corresponds to the final energy accumulated by the spring, and another linear term in the displacement, which describes the way in which energy accumulates during the compression process, when there is a precondition of energy accumulation due to the initial deformation. It should be noted that the integrand of Equation (8) corresponds to the data provided, an initial deformation δ and an instantaneous deformation, described by the auxiliary variable *r*, which acts while the body is attached to the spring.

Therefore, substituting the above into equation (6), it results,

$$-\frac{1}{2}mv_{i2}^2 - mgx(sin\alpha) = -mg\mu_f x(cos\alpha) - \frac{1}{2}kx^2 - k\delta x \quad (9)$$

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From equation (4), it is obtained,

$$\frac{1}{2}mv_{f1}^2 = \frac{1}{2}mv_{i1}^2 + mgd(sin\alpha) - dmg\mu_f(cos\alpha) \quad (10)$$

In position 2 it is fulfilled that, $v_{i2}=v_{f1}$, substituting in the expression of the previous energy balance, given by Equation (9) results,

$$-\frac{1}{2}mv_{i1}^2 - mgd(sin\alpha) + dmg\mu_f(cos\alpha) - mgx(sin\alpha) = -mg\mu_f x(cos\alpha) - \frac{1}{2}kx^2 - k\delta x \quad (11)$$

Simplifying Equation (11), it results

$$-\frac{1}{2}mv_{i1}^2 - mgd(sin\alpha) + dmg\mu_f(cos\alpha) - mgx(sin\alpha) + mg\mu_f x(cos\alpha) + \frac{1}{2}kx^2 + k\delta x = 0 \quad (12)$$

Factoring Equation (12) we obtain,

$$kx^2 + 2\big(k\delta + \mu_f mg(cos\alpha) - mg(sin\alpha)\big)x - mv_{1i}^2 - 2\big(mgd(sin\alpha) - dmg\mu_f(cos\alpha)\big) = 0 \quad (13)$$

Some interesting results can be raised by observing the form of Equation (13), such as the proportion that exists in the coefficients of the polynomial in the displacement variable *x*, and the possibility of designing a particular behaviour, but most importantly in this paper it is to observe that the solution provided by the principle of conservation of mechanical energy and work produces a second order polynomial in the variable *x* that admits two real roots.

The use of the principle of conservation of mechanical energy requires the concept of work, particularly to determine the energetic effect of the force of friction, so that in problems that include friction this perspective will be used, which is recommendable in teaching the principle of conservation of mechanical energy, that is, in its most general form, in design problems the principle of conservation of mechanical energy can be enriched with the concept and application of work, and although this is developed in textbooks, it is important to emphasize the reciprocity of both concepts.

IV. SOLUTION BY NEWTON'S EQUATIONS

Newton's formulation is made for the second stage in which the body comes into contact with the spring, δ corresponds to the previous deformation at which the spring was initially found, according to the parameters of the problem. In this analysis, the identified Equation is,

$$\sum F = m\ddot{x} \quad (14)$$

Where,

$$\sum F = -k(x+\delta) + mg(sin\alpha) + \mu N \quad (15)$$

Resulting in the following differential equation,

$$m\ddot{x} + kx = -k\delta + mg(sin\alpha) + \mu N \quad (16)$$

Along with the following initial conditions,

$$x(t=0) = 0 \quad (17)$$

$$\dot{x}(t=0) = v_{f1} = v_{i2} \quad (18)$$

The solution of this differential equation together with its initial conditions is,

$$x(t) = \left(\delta - \frac{mg(sin\alpha)}{k} - \frac{\mu N}{k}\right)cos\omega_n t + \frac{v_{f1}}{\omega_n}sin\omega_n t + \frac{mg(sin\alpha)}{k} - \delta + \frac{\mu N}{k} \quad (19)$$

The instant time for which the maximum displacement occurs is,

$$t_{max} = \frac{1}{\omega_n}arctan\frac{v_{f1}}{\omega_n\left(\delta - \frac{mg(sin\alpha)}{k} - \frac{\mu N}{k}\right)} \quad (20)$$

This is obtained by deriving the position expression in time and equalling it to zero. If the expression of the previous time is evaluated, Equation (19), the position turns out to be,

$$x_{max} = x(t_{max}) = \left(\frac{mg(sin\alpha)}{k} - \frac{\mu N}{k} - \delta\right) + \sqrt{\left(\frac{mg(sin\alpha)}{k} - \frac{\mu N}{k} - \delta\right)^2 + \left(\frac{v_{f1}}{\omega_n}\right)^2} \quad (21)$$

Equation (19) contains two harmonic terms of the same frequency but different amplitude, so it can be expressed by means of a single trigonometric function. Expressing equation (19) in terms of the function $cos\,\omega_n t$, it results in the following position function,

$$x(t) = \sqrt{\left(\frac{mg(sin\alpha)}{k} - \frac{\mu N}{k} - \delta\right)^2 + \left(\frac{v_{f1}}{\omega_n}\right)^2} \cdot cos\left(\omega_n t - arctan\left(\frac{\frac{v_{f1}}{\omega_n}}{\left(\frac{mg(sin\alpha)}{k} - \frac{\mu N}{k} - \delta\right)}\right)\right) + \left(\frac{mg(sin\alpha)}{k} - \delta - \frac{\mu N}{k}\right) \quad (22)$$

When evaluating the Equation (22), for the time of maximum displacement, given by Equation (20), it results,

$$x_{max_1} = \left(\frac{mg(sin\alpha)}{k} - \frac{\mu N}{k} - \delta\right) + \sqrt{\left(\frac{mg(sin\alpha)}{k} - \frac{\mu N}{k} - \delta\right)^2 + \left(\frac{v_{f1}}{\omega_n}\right)^2} \quad (23)$$

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Now it is observed that the previous expression corresponds to the solution of a quadratic equation of the type,

$$ax^2 + bx + c = 0 \quad (24)$$

For the coefficients,

$$a = \frac{1}{2}, b = \left(\frac{mg(sin\alpha)}{k} - \frac{\mu N}{k} - \delta\right), c = \frac{1}{2}\left(\frac{v_{f1}}{\omega_n}\right)^2 \quad (25)$$

Substituting into equation (24), it results,

$$\frac{1}{2}x^2 - \left(\frac{mg(sin\alpha)}{k} - \frac{\mu N}{k} - \delta\right)x - \frac{1}{2}\left(\frac{v_{f1}}{\omega_n}\right)^2 = 0 \quad (26)$$

Which can be developed to write,

$$\frac{1}{2}kx^2 - (mg(sin\alpha) - \mu N - k\delta)x - \frac{1}{2}m{v_{f1}}^2 = 0 \quad (27)$$

Remembering Equation (4), and taking into account that,

$${\omega_n}^2 = \frac{k}{m} \quad (28)$$

It has been shown that the quadratic Equation (24) corresponds to the equation determined by the theorem of work and kinetic energy and potential energy, Equation (13).

The next point of maximum amplitude corresponds to the half period of time of the oscillation of the mass, that is,

$$t_{max_2} = \frac{1}{\omega_n} arctan \frac{v_{f1}}{\omega_n\left(\delta - \frac{mg(sin\alpha)}{k} - \frac{\mu N}{k}\right)} + \frac{\pi}{\omega_n} \quad (29)$$

Then we obtain the expression of the position,

$$x_{max_2} = -\sqrt{\left(\frac{mg(sin\alpha)}{k} - \frac{\mu N}{k} - \delta\right)^2 + \left(\frac{v_{f1}}{\omega_n}\right)^2} + \left(\frac{mg(sin\alpha)}{k} - \frac{\mu N}{k} - \delta\right) \quad (30)$$

Which corresponds to the second root of the quadratic equation, so it has been shown that the second solution of the quadratic equation is obtained by the principle of conservation of energy? It is concluded that the second root corresponds to the return position of the mass once the spring has been fully compressed.

It was necessary to solve the problem in the first place with the principle of conservation of energy and work to arrive at the quadratic equation that presents two real solutions, one positive and one negative but both real, later it was necessary to solve the problem in two stages to identify that the quadratic equation has within its coefficients the final speed with which the body comes into contact with the mass to determine the previously obtained quadratic equation, and finally the problem has been resolved from Newton's perspective to demonstrate that the second root of the quadratic equation is a term that represents a real solution, which is normally discarded because it does not make a physical sense, which is an error, because it corresponds to the maximum position once the mass is in your return position.

V. EXAMPLE

Consider the following numerical values for the problem initially posed, in Table 1.

Table 1: Dynamic data for the mechanical system

d	9 m
k	20,000 N/m
m	55 kg
δ	0.06m
μ	0.2
α	20º
$\boldsymbol{v_{i1}}$	2.0 m/s

According to the previous process, it is determined from equation (13), the second order polynomial for the position x,

$$20000x^2 + 2233.729638x - 1716.43327 = 0 \quad (31)$$

The roots of this polynomial are; x_1 = 0.24238m, x_2 = -0.35407m. On the other hand, solving the differential Equation (16) numerically, with the Runge-Kutta-Fehlberg 4/5 method and together with the initial conditions (17) and (18), the following graphic solution is obtained,

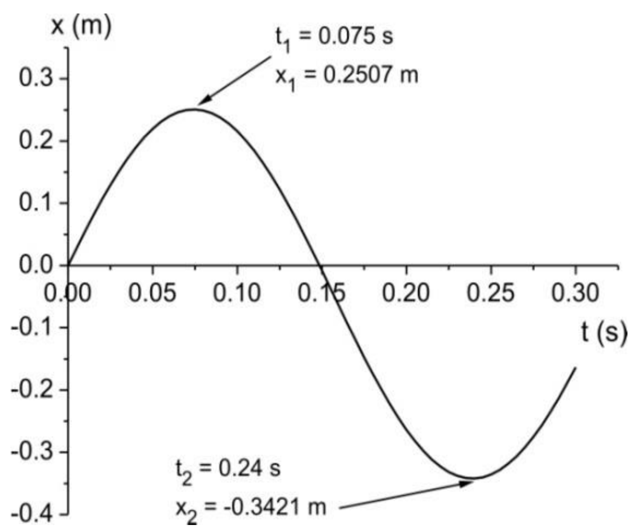

Figure 4: Time response of the body while in contact with the spring

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From the graph of Figure 4, the following values are observed, t_1 = 0.075sec, x_1 = 0.2507m, t_2 = 0.24sec, x_2 = -0.3421m.

The time values for the maximum displacements calculated with expressions (20) and (29) are, t_{max1} = 0.0743sec, t_{max2} = 0.239 sec.

The previous values of time correspond to the maximum displacements of the body m, which verifies that the negative root makes a physical sense, provided by the second root of the polynomial resulting from the energy balance.

VI. CONCLUSION

In this work, the solution to a well-known problem in the teaching of the energy method for the basic courses of physics and engineering has been developed. The solution is developed by two different strategies, using the principle of conservation of energy and the theorem of kinetic energy and work, and by means of Newton's second law, in both cases the same result is reached.

In the solution based on the principle of conservation of energy, the strategy consists of evaluating the energy changes in two stages, defined by three positions, this allows us to identify and assess the way in which the energy is transformed, it is possible to determine the intermediate speed between the stage of point 2 and point 3. Also, the definition of stages allows establishing the validity of the action of friction, which in turn is useful to define positions to quantify the change of gravitational potential energy. In the case of the solution by means of Newton's second law, the corresponding differential equation is obtained, together with the initial conditions of the problem.

The differential equation obtained is valid in the region in which the effects of gravity, friction and elastic act together, which in turn includes the previous compression and the speed condition at which the body comes into contact with the spring. The second degree polynomial is an expression of displacement. In this problem, the body travels a distance twice times, that is, it follows a trajectory in two directions, so the polynomial is associated with the movement of the body, although the effect of displacement in two directions is produced by the elastic action of the spring, the two solutions provided by the second degree polynomial are physically real. From the energetic point of view, there are two stages of exchange of potential energy to kinetic energy.

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