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Generating Symmetric Group Representations for Network Dynamics and Groupoid Formalism

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***Abstract* - Understanding the concept of group theory and to apply it in other field of sciences has been a problem among undergraduate students. Although there are some attempt by Dubinsky et al, [1], where some finite groups were discussed among students, the paper was limited to theoretical aspect of the topic. This research is therefore designed to explore clearly the procedure for constructing finite groups for better understanding of the subject area in the given domain. The research is channeled towards the use of group theory in Network Dynamics, which serves as concrete application of finite groups to the students. Some minor open problems with regards to algebraic graph theory was discussed which lead to network dynamics and groupoid formalism.**

Keywords: Finite groups, Subgroups, Transformations, Representations, Homomorphism, Isomorphism, Network Dynamics.

I. INTRODUCTION

The idea of Group arises in mathematics as "sets of symmetries (of an object), which are closed under composition and inverses". A concrete example is the Symmetric group S_n whose elements consists of all possible permutations of n -objects; the group of even permutations in S_n called Alternating group A_n; the Dihedral group D_{2n} (also called geometric group) which is the group of symmetries of regular n-gons in the plane; the Orthogonal group called the group of distance-preserving transformations in the Euclidean space that fixes the origin. Generally, from geometric point of view, questions such as "Given a geometric object X, what is its group of symmetries?" aroused while the same question is reversed in Representation theory such as "Given a group G, what objects X does it act on?" and the attempt to answer such question leads to the classification of X up to isomorphism. Group theory is a branch of abstract algebra developed to study and manipulate abstract concepts involving symmetry [2]. It also serves as a tool to obtain information on various molecular structures with applications to many areas of sciences such as signal processing, image processing, cryptography, sound compression among others [3], [4], and Pitch analysis [5]. Furthermore, abstract algebra presents a serious educational problem where Mathematics students consider it as one of the most troublesome undergraduate subjects. It appears to give students a great difficulty both in terms of dealing with the content and the actual visualization of its concrete applications.

In many institutions, abstract algebra is one of the first courses for students in which they must come to grips with abstract concepts and to work with some important mathematical principles and proofs. Although there are no formal studies but it is known that many students after taking this course, usually turn off from abstract mathematics. Since a significant percentage of the student audience for abstract algebra consists of future mathematics teachers, it is particularly important that the profession of mathematics education develop effective pedagogical strategies for improving the attitude of high school mathematics teachers towards mathematical abstraction [1]. Another good reason that is related to abstraction for the importance of abstract algebra in general, is that an individual's knowledge of the concept of group should include an understanding of various mathematical properties and constructions independent of particular examples. This research is designed to present and describe how to generate, construct and classify groups based on their symmetric properties and actions on sets. The symmetric properties to be generated can be used for encryption-decryption analysis and in signal processing. Other literatures such as "The abstract harmonic analysis on finite, non-abelian groups" as discussed by Knapp, [3], plays a very important role due to the importance of spectral techniques and Fourier analysis in the consideration of various classes of functions [6]. In particular, the basic interest and approach in this research is to focus on transfer of some important results from geometric structures on the real line to algebraic structures so as to generate more properties to be used in network dynamics. Among many groups, it is found that finite non-abelian groups have more and concrete applications in different areas of science and engineering. Hence, the groups Sn and Dn will be used for demonstration purposes.

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1.1 Justification and Objectives

The research is designed to address the existing problem among undergraduate mathematics students on how to understand the concept of group theory in more concrete way and how to construct, generate and manipulate groups of different order.

The research objectives are to:

i. Simplify how individuals can learn specific topics in group theory,
ii. Develop some new methods for constructing groups of symmetries,
iii. Present a concrete application of symmetric group theory.

II. REVIEW OF RELEVANT WORK

2.1 Group and Symmetries

Group Theory addresses the way in which certain collections of mathematical "objects" are related to each other [7]. For example, the set Z of integers constitute a group because under certain conditions (particularly, addition), the relationships between the integers obey the rules of group. Group theory is also the mathematical application of symmetry to an object to obtain knowledge of its physical properties [8]. What group theory brings to the table is how the symmetry of molecule is related to its physical properties and provides a quick simple method to determine the relevant physical information of the molecule. The symmetry of a molecule provides us with the information of what energy levels the orbital will be, what the orbital symmetries are, what transitions can occur between energy levels, even bond order can be found, all without rigorous calculations. The fact that so many important physical aspects can be derived from symmetry is a very profound statement and this is what makes group theory so powerful [8]. A good example of symmetry in real life is the mirror symmetry. Our bodies have, to a good approximation, mirror symmetry in which our right side is matched by our left as if a mirror passed along the central axis of our bodies. Our legs and hands illustrates this most vividly, so much so that the image is carried over to some studies in chemistry and biology when one speaks of a molecule as being either "left" or "right" handed. Algebraically, an object is described as symmetric with respect to a given transformation if the object appears to be in a state, that is, identical to its initial state, after the transformation. In geometry, most types of symmetry can be described in terms of an apparent movement of the object, such as some types of rotation or translation. The apparent movement is called the symmetry operation. The locations where the symmetry operations occur such as a rotation axis, a mirror plane, an inversion center, or a translation vector are described as symmetry elements. There are many other examples of mirror symmetry in ordinary life. We can also see more complex symmetry in the pattern around us. It can be found in floor-tiles arrays, cloth designs, flowers and mineral crystals. The basic mathematics of symmetry also applies to music [9], dance (square dance) and even the operations require in solving Rubik's cube. The rules that govern symmetry are found in the mathematics of group theory.

2.2 Finite Groups

A group is called *finite* if it has a finite number of elements. The number of elements is called the order of the group [8]. An important class is the Symmetric group S_n, the groups of permutations of n letters. For example, the symmetric group on 3 letters S3 is the group consisting of all possible swaps of the three letters *ABC*, i.e., contains the elements *ABC, ACB, ..., CBA*, in total of 6 (or 3!) elements. This class is fundamental as any finite group can be expressed as a subgroup of a symmetric group S_n for a suitable integer n. *The order of an element x in a group G is the least positive integer n for which* $x^n = e$, *where* x^n *represents* $x\bullet x\bullet x\bullet \ldots \bullet x$ *(n-times) i.e. application of the operation • to n copies of x (If • represents multiplication, then* x^n *corresponds to the* n^{th} *power of x). In infinite groups, such an n may not exist, in which case the order of x is said to be infinity. The order of an element equals the order of the cyclic subgroup generated by this element [2].*

2.3 Group Action and Arbitrary Transformation

According to Samaila[5], representation of a group is given as a function ρ from a group G to the general linear group of some vector space V such that ρ is a homomorphism. Some measures on how closed an arbitrary function defined between non trivial groups is to being a homomorphism were also presented by [10]. It was shown that the functions exhibit some properties which are similar to that of conjugates and commutators. The authors also showed that there exists a theory based on the given structures and the same theory was used in unifying some approach like group cohomology and transfer. Instead of trying to prove that some homomorphism exist especially by orbit counting, they tried to build using direct approach which means to begin with any given arbitrary function, then average out the same function action. For example, given a finite group G and H≤G such that [G:H] = r. Define a homomorphism $\rho:H\to K$ from H to an Abelian group K and a collection of all cosets representatives $\{\alpha_i : 1 \le i \le r\}$ of H in G for the cosets $\{\alpha_i H\}$. Then a function σ was defined by $\sigma:G\to K$ such that $\sigma(h\alpha_i) = \rho(h)$. It was also shown that if $h_o \in H$, then

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$$\sigma^{h_0}(h\alpha_i) = \sigma(h_0)^{-1}\sigma(h_0 h\alpha_i) = \rho(h_0)^{-1}\rho(h_0 h) = \rho(h) = \sigma(h\alpha_i)$$

So that H stabilized σ under the function action. Since the transfer function is a power of the average function, it was concluded that it is a homomorphism. The function action also measures the failure of a function to be a homomorphism in the same way that the conjugation action measures a failure to commute. A construction analogous to commutators, called distributor which also measures the extent to which an arbitrary function preserves group structure was also presented by [10]. The set of distributors given as x and y range over a group G. It measures the extent to which a function ρ fails to be a homomorphism and also measures the extent to which the function ρ distributes over group of multiplication. From geometric point of view, the above properties of group action and arbitrary functions can be used to manipulate functions from one form to another. Any change to the function may be seen as a change of every point (the domain) to another point (the range) [11]. The pairwise correspondence of points in the domain and range is pictured as a mapping. When the mapping is injective, then it is a transformation, which means every point in the domain corresponds to only one point (image) in the range. Transformations do not reduce the size of an object. Therefore, there is always an inverse transformation to bring back the object to the original domain. There is also an identity transformation which does not bring any change to an object by mapping point to the same point in the domain itself. Further, if two transformations are applied to an object consecutively, then it is possible to combine the effect of both in a single transformation. A special case of transformations is linear transformations in n-dimensional space. In a linear transformation, with respect to a fixed coordinate system, a point with coordinates (v_1, v_2, …,v_n) is mapped to a point with coordinate $(v_1', v_2', ..., v_n')$ where the v_i' coordinates are given by:

$$v_i' = \sum_{j=1}^{n} a_{ij} v_i \qquad i = 1, 2, \ldots, n.$$

The n^2 coefficients a_{ij} form a two-dimensional array T_i, which is the matrix of the transformation for all points $v_i$to v_i'. The transformation is invertible if the matrix is invertible. Again, an identity and inverse transformation matrix may be defined in the n-dimensional space for points v_i.

Some analysis was performed with finite dimensional groups by Huang, [12]. The dihedral transformation group contains rotations and reflections and the number of irreducible representations of a transformation group is equal to the number of conjugacy classes by Schur's Lemma. He then showed that for dihedral groups G,if 'n' is the dimension of the space, then the number of irreducible representations is given by (n+6)/2 for even sized 'n' and (n+3)/2 for odd sized 'n'. Hence this number i.e., the number of conjugacy classes, becomes the actual dimension of the space.

2.4 Group Representations

The goal of group representation theory is to study groups via their actions on vector spaces. Consideration of groups acting on sets leads to such important results as the Sylow theorems. By acting on vector spaces even more detailed information about a group can be obtained. This is the subject of representation theory. As byproducts, emerged Fourier analysis on finite groups and the study of complex-valued functions on a group.

If $\phi : Z_n \to C$ and $\varphi : Z_n \to C$ are representations on Z_n defined by $\phi_m = e^{\frac{2\pi i m}{n}}$ and $\varphi_m = e^{-\frac{2\pi i m}{n}}$ respectively, then we define the sum $\phi \oplus \varphi$ by

$$(\phi \oplus \varphi)_m = \begin{pmatrix} e^{\frac{2\pi m i}{n}} & 0 \\ 0 & e^{\frac{-2\pi m i}{n}} \end{pmatrix}.$$

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Since representations are special kind of homomorphism, if a group G is generated by a set X, then a representation ϕ of G is determined by its values on X; of course, not any assignment of matrices to the generators gives a valid representation [13].

Let ϕ: $G \rightarrow GL(V)$ be a representation. If $W \leq V$ is a G-invariant subspace, we may restrict ϕ to obtain a representation $\phi|_W$:$G \rightarrow GL(W)$ by setting $(\phi|_W)_g(w) = \phi_g(w)$ for $w \in W$. Precisely because W is G-invariant, we have $\phi_g(w) \in W$ and $\phi|_W$ is sometimes called a *subrepresentation*of ϕ. Any degree one representation ϕ: $G \rightarrow C$is irreducible, since Chas no proper non-zero subspaces. If $G = \{1\}$ is the trivial group and ϕ: $G \rightarrow GL(V)$ is a representation, then necessarily $\phi_1 = e$. Thus, a representation of the trivial group has the same properties as a vector space. For the trivial group, a G-invariant subspace is nothing more than a subspace and a representation of $\{1\}$ is irreducible if and only if it has degree one.

Now, for two-dimensional representations, let ρ: $S_3 \rightarrow GL_2(C)$ be a representation specified on the generators (1 2) and (1 2 3) by

$$\rho_{(12)} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \ \rho_{(123)} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

and let δ:$S_3 \rightarrow C$ be defined by $\delta_\sigma = 1$. Then

$$(\rho \oplus \delta)_{(12)} = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{and } (\rho \oplus \delta)_{(123)} = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

According to Benjamin, [13], on representation of finite groups, if ϕ: $G \rightarrow GL(V)$ is a representation of degree 2 (i.e., $dimV = 2$), then ϕ is irreducible if and only if there is no common eigenvector v to all $\phi_g$with $g \in G$.

This method of using eigenvectors only works for degree 2 representations.

If $T : V \rightarrow V$ is a linear transformation and B is a basis for V, Benjamin, [13] used $[T]_B$ to denote the matrix for T in the basis B. Let ϕ: $G \rightarrow GL(V)$ be a decomposable representation, say $V = V_1 \oplus V_2$ where V_1,V_2 are non-trivial G-invariant subspaces and $\phi^{(i)} = \phi_{Vi}$. Choose bases B_1 and B_2 for V_1 and V_2, respectively. Then it follows from the definition of a direct sum that $B = B_1 \cup B_2$ is a basis for V. Since V_i is G-invariant, then $\phi_g(B_i) \subseteq V_i = CB_i$. Hence,

$$[\phi_g]_B = \begin{pmatrix} [\phi^{(1)}]_{B_1} & 0 \\ 0 & [\phi^{(2)}]_{B_2} \end{pmatrix}$$

So that $\phi \sim \phi^{(1)} \oplus \phi^{(2)}$.

Complete reducibility is the analogue of diagonalizability in representation theory. The goal is to show that any representation of a finite group is completely reducible. This can be achieved by showing that any representation is either irreducible or decomposable, and then proceed by induction on the degree.

Now based on the result of Benjamin 2009, if ϕ: $G \rightarrow GL(V)$ is equivalent to a decomposable representation, then ϕ is decomposable.

To see this, let ψ: $G \rightarrow GL(W)$ be a decomposable representation with $\psi \sim \phi$ and $T : V \rightarrow W$ a vector space isomorphism with $\phi_g = T^{-1}\psi_g T$. Suppose that W_1 and W_2 are non-zero invariant subspaces of W with $W = W_1 \oplus W_2$. Since T is an equivalence, the functions ϕ_g: $V \rightarrow V$ (also ϕ_g: $W \rightarrow W$) and $T : V \rightarrow W$ commutes, i.e., $T\phi_g = \psi_g T$ for all $g \in G$. Let $V_1 = T^{-1}(W_1)$ and $V_2 = T^{-1}(W_2)$. First we claim that $V = V_1 \oplus V_2$. Indeed, if $v \in V_1 \cap V_2$, then $Tv \in W_1 \cap W_2 = \{0\}$ and so $Tv = 0$. But T is injective so this implies $v = 0$.

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Next, if $v \in V$, then $Tv = w_1 + w_2$ for some $w_1 \in W_1$ and $w_2 \in W_2$, and $v = T^{-1}w_1 + T^{-1}w_2 \in V_1 + V_2$. Hence, $V = V_1 \oplus V_2$.

Again to show that V_1 and V_2 are G-invariant, if $v \in V_i$, then $\phi_g v = T^{-1}\psi_g Tv$. But $Tv \in W_i$ implies $\psi_g Tv \in W_i$ since W_i is G-invariant. Therefore, we conclude that $\phi_g v = T^{-1}\psi_g Tv \in T^{-1}(W_i) = V_i$, as required.

Similarly, if ϕ: $G \to GL(V)$ is equivalent to an irreducible representation, then ϕ is irreducible and if ϕ: $G \to GL(V)$ is equivalent to a completely reducible representation, then ϕ is completely reducible.

It is therefore deduced from the above results that if V is an inner product space, then a representation ϕ: $G \to GL(V)$ is called *unitary* if $\phi_g$is unitary for all $g \in G$, i.e., $\langle \phi_g(v), \phi_g(w) \rangle = \langle v, w \rangle$ for all $v, w \in W$. In other words, ϕ: $G \to U(V)$. Identifying $GL_1(C)$ with C, we see that a complex number z is unitary if and only if $\bar{z} = z^{-1}$, that is $\bar{z}z = 1$. But this says exactly that $|z| = 1$, so $U_1(C)$ is exactly the unit circle S^1 in C. Hence a one-dimensional unitary representation is a homomorphism ϕ: $G \to S^1$.

A simple example is a function ϕ: $R \to S^1$ defined by $\phi(t) = e^{2\pi it}$. Then ϕ is a unitary representation of Rsince $\phi(t + s) = e^{2\pi i(t+s)} = e^{2\pi it}e^{2\pi is} = \phi(t)\phi(s)$ and produced a symmetric signal as follows:

```
>> t = linspace(0,10,20);
>> Y = (exp(2*pi*i*t));
>>stem(t,Y,'filled')
```

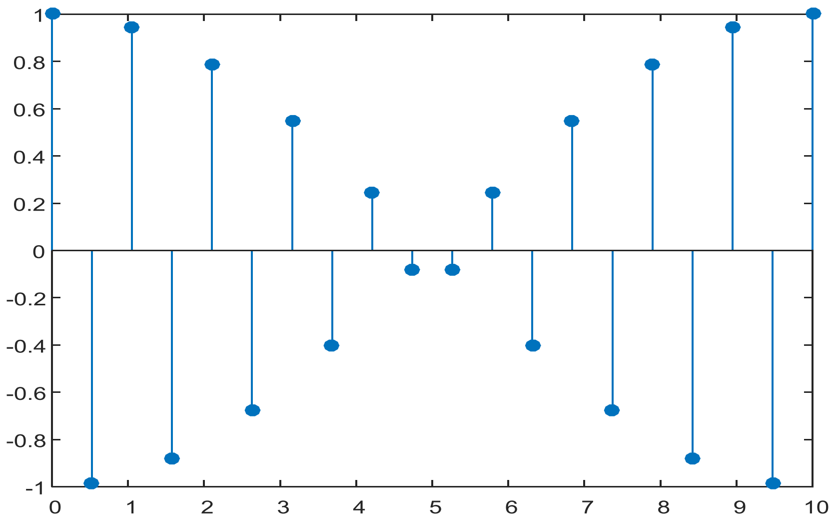

Figure 2.1: Symmetric signal produced by a Unitary representation

It turns out that every representation of a finite group G is equivalent to a unitary representation.

III. METHODOLOGY

3.1 Location of the Study Area

The research was carried out in Adamawa State, Nigeria, bordered by the states of Borno to the northwest, Gombe to the west and Taraba to the southwest. Its eastern border forms the national eastern border with Cameroon. Topographically, it is a mountainous land crossed by the large river valleys – Benue, Gongola and Yedsarem. The institutions involved are Modibo Adama University (MAU), Yola, Adamawa State which is situated between Latitude 9^0 20' 00" and 9^0 21' 30" N and Longitude 12^0 29' 00 "E and 12^0 30’ 30"E, and Adamawa State University (ADSU), Mubi, Adamawa State which is situated between Latitude 10° 16' 8" North, Longitude 13° 16' 14" East.

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3.2 Specific Subject Areas

3.2.1 Properties of Groups

Group features such as order of the group, order of an element in a group, cyclic group, group action and morphisms in a group were presented in a simplified version. The basic ideas in this case are to make the students visualize group as a generic object. This involves steps in the construction of single group which begins with the realization that other groups may be constructed from known groups. At this point, a developing conception of isomorphism was involved and the students developed the process of forming several groups and establishing isomorphisms between them.

3.2.2 Coset and Normality

In this case, a mathematical method for construction of cosets was developed. This was used to construct a binary operation on the set of cosets to obtain a new group, called quotient group. But this construction of a new group may not actually work in all cases because it depends on normality which is a property that a subgroup may or may not have any relation with the group. The logic involved in these processes was extremely difficult for students, including constructing the concept of normality.

3.2.3 Network Dynamics

In the aspect of network dynamics, we used some methods of describing group theory formalism applied to graphs, and then step it up in symmetry and describe the algebraic object called groupoid. This will therefore bring more dynamics into the study [14]. According to [1], it is known that a directed graph encodes the dynamics given by $\frac{dx_i}{dt} = f_i(A, x_i)$ where x_i is the state of molecular species i, and A_i is the full interactome adjacency matrix. More precisely the automorphism group of the network implicitly encodes the dynamics. But Groupoids are algebraic objects that resemble groups where the conventional group operation is not defined. Hence, we recognized symmetry since automorphisms will be nontrivial. This formalism will therefore allow us to apply symmetric group operations to network graphs.

3.3 GAP: Groups, Algorithms, Programming - a System for Computational Discrete Algebra

GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory. GAP provides a programming language, a library of thousands of functions implementing algebraic algorithms written in the GAP language as well as large data libraries of algebraic objects. This group application package was used to construct some groups and morphisms between finite groups.

IV. RESULTS

This research aimed at simplifying the properties of finite groups and its representations. The groups S_n and D_n were used, where $S_5$is used for demonstration purpose. Few subgroups were derived in more simplified version. Let G = S_5. Then the one-headed group Gis the group of permutations of the set $S = \{1, 2, 3, 4, 5\}$, i.e., the set of all bijections $\sigma : S \rightarrow S$ defined by $\sigma(a_i) = a_j; i, j \leq 5$. The collection of all such bijections give rise to a group of order 120 as follows:

$$G = \{i, \rho_1, \rho_2, ..., \rho_{10}, \sigma_1, \sigma_2, ..., \sigma_{20}, \tau_1, \tau_2, ..., \tau_{30}, \gamma_1, \gamma_2, ..., \gamma_{15}, \beta_1, \beta_2, ..., \beta_{24}, \delta_1, \delta_2, ..., \delta_{20}\}$$

Where i = (1);

ρ_1 = (4 5), ρ_2 = (3 5), ρ_3 = (3 4), ρ_4 = (2 5), ρ_5 = (2 3), ρ_6 = (2 4), ρ_7 = (1 5), ρ_8 = (1 4), ρ_9 = (1 3), ρ_{10} = (1 2);

σ_1 = (1 2 3), σ_2 = (1 3 2), σ_3 = (1 2 4), σ_4 = (1 4 2), σ_5 = (1 2 5), σ_6 = (1 5 2), σ_7 = (1 3 4), σ_8 = (1 4 3), σ_9 = (1 4 5), σ_{10} = (1 5 4), σ_{11} = (1 3 5), σ_{12} = (1 5 3), σ_{13} = (2 3 4), σ_{14} = (2 4 3), σ_{15} = (2 3 5), σ_{16} = (2 5 3), σ_{17} = (2 4 5), σ_{18} = (2 5 4), σ_{19} = (3 4 5), σ_{20} = (3 5 4);

τ_1 = (2 3 4 5), τ_2 = (2 5 4 3), τ_3 = (2 3 5 4), τ_4 = (2 4 5 3), τ_5 = (2 4 3 5), τ_6 = (2 5 3 4), τ_7 = (1 2 3 4), τ_8 = (1 4 3 2), τ_9 = (1 2 3 5),

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τ_{10} = (1 5 3 2), τ_{11} = (1 2 4 3), τ_{12} = (1 3 4 2), τ_{13} = (1 2 4 5), τ_{14} = (1 5 4 2), τ_{15} = (1 2 5 3), τ_{16} = (1 3 5 2), τ_{17} = (1 2 5 4), τ_{18} = (1 4 5 2), τ_{19} = (1 3 4 5), τ_{20} = (1 5 4 3), τ_{21} = (1 3 5 4), τ_{22} = (1 4 5 3), τ_{23} = (1 3 2 4), τ_{24} = (1 4 2 3), τ_{25} = (1 3 2 5), τ_{26} = (1 5 2 3), τ_{27} = (1 4 3 5), τ_{28} = (1 5 3 4), τ_{29} = (1 4 2 5), τ_{30} = (1 5 2 4);

γ_1 = (2 4)(3 5), γ_2 = (2 5)(3 4), γ_3 = (2 3)(4 5), γ_4 = (1 3)(2 4), γ_5 = (1 3)(2 5), γ_6 = (1 4)(2 3), γ_7 = (1 4)(2 5), γ_8 = (1 5)(2 3), γ_9 = (1 5)(2 4), γ_{10} = (1 4)(3 5), γ_{11} = (1 5)(3 4), γ_{12} = (1 2)(3 4), γ_{13} = (1 2)(3 5), γ_{14} = (1 3)(4 5), γ_{15} = (1 2)(4 5);

β_1 = (1 2 3 4 5), β_2 = (1 3 5 2 4), β_3 = (1 4 2 5 3), β_4 = (1 5 4 3 2), β_5 = (1 2 3 5 4), β_6 = (1 3 4 2 5), β_7 = (1 5 2 4 3), β_8 = (1 4 5 3 2), β_9 = (1 2 4 5 3), β_{10} = (1 4 3 2 5), β_{11} = (1 5 2 3 4), β_{12} = (1 3 5 4 2), β_{13} = (1 2 4 3 5), β_{14} = (1 4 5 2 3), β_{15} = (1 3 2 5 4), β_{16} = (1 5 3 4 2), β_{17} = (1 2 5 4 3), β_{18} = (1 5 3 2 4), β_{19} = (1 4 2 3 5), β_{20} = (1 3 4 5 2), β_{21} = (1 2 5 3 4), β_{22} = (1 5 4 2 3), β_{23} = (1 3 2 4 5), β_{24} = (1 4 3 5 2);

δ_1 = (1 2 3)(4 5), δ_2 = (1 3 2)(4 5), δ_3 = (1 2 4)(3 5), δ_4 = (1 4 2)(3 5), δ_5 = (1 2 5)(3 4), δ_6 = (1 5 2)(4 5), δ_7 = (1 3 4)(2 5), δ_8 = (1 4 3)(2 5), δ_9 = (1 4 5)(2 3), δ_{10} = (1 5 4)(2 3), δ_{11} = (1 3 5)(2 4), δ_{12} = (1 5 3)(2 4), δ_{13} = (1 5)(2 3 4), δ_{14} = (1 5)(2 4 3), δ_{15} = (1 4)(2 3 5), δ_{16} = (1 4)(2 5 3), δ_{17} = (1 3)(2 4 5), δ_{18} = (1 3)(2 5 4), δ_{19} = (1 2)(3 4 5), δ_{20} = (1 2)(3 5 4);

Recall that the order of an element x of a group G is the least positive integer m for which $x^m = e$, the identity element of G, where x^m represents x·x·x· … ·xm-times. Then we classify the elements of G according to their order as follows:

Table 4.1: Order of elements of G

Order	Elements	Formula Calculating Element Order
1	*I*	LCM{1}
2	$\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7, \rho_8, \rho_9, \rho_{10}, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{15}$	LCM{2,1}
3	$\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}, \sigma_{20}$	LCM{3,1}
4	$\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6, \tau_7, \tau_8, \tau_9, \tau_{10}, \tau_{11}, \tau_{12}, \tau_{13}, \tau_{14}, \tau_{15}, \tau_{16}, \tau_{17}, \tau_{18}, \tau_{19}, \tau_{20}, \tau_{21}, \tau_{22}, \tau_{23}, \tau_{24}, \tau_{25}, \tau_{26}, \tau_{27}, \tau_{28}, \tau_{29}, \tau_{30}$	LCM{4,1}
5	$\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8, \beta_9, \beta_{10}, \beta_{11}, \beta_{12}, \beta_{13}, \beta_{14}, \beta_{15}, \beta_{16}, \beta_{17}, \beta_{18}, \beta_{19}, \beta_{20}, \beta_{21}, \beta_{22}, \beta_{23}, \beta_{24}$	LCM{5,1}
6	$\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \delta_{10}, \delta_{11}, \delta_{12}, \delta_{13}, \delta_{14}, \delta_{15}, \delta_{16}, \delta_{17}, \delta_{18}, \delta_{19}, \delta_{20}$	LCM{2,3}

4.1 Subgroups of order 8

Since the factors of 8 are 1, 2, 4 and 8, elements of the subgroup of order 8 must have orders 2 or 4, except the identity element. Consider the following set of elements

Q = {*i*, (2 3 4 5), (2 5 4 3), (2 4)(3 5), (2 4), (3 5), (2 3)(4 5), (2 5)(3 4)} = {*i*, τ_1, τ_2, γ_1, ρ_6, ρ_2, γ_3, γ_2}

Then this is a subgroup of G of order 8. To see this, we construct a group table for Q as follows:

Table 4.2: Group table of Q

*	*i*	τ_1	τ_2	γ_1	ρ_6	ρ_2	γ_3	γ_2
i	*i*	τ_1	τ_2	γ_1	ρ_6	ρ_2	γ_3	γ_2
τ_1	τ_1	γ_1	*i*	τ_2	γ_3	γ_2	ρ_2	ρ_6

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τ_2	τ_2	i	γ_1	τ_1	γ_2	γ_3	ρ_6	ρ_2
γ_1	γ_1	τ_2	τ_1	i	ρ_2	ρ_6	γ_2	γ_3
ρ_6	ρ_6	γ_2	γ_3	ρ_2	i	γ_1	τ_2	τ_1
ρ_2	ρ_2	γ_3	γ_2	ρ_6	γ_1	i	τ_1	τ_2
γ_3	γ_3	ρ_6	ρ_2	γ_2	τ_1	τ_2	I	γ_1
γ_2	γ_2	ρ_2	ρ_6	γ_3	τ_2	τ_1	γ_1	i

Clearly, from Table 2 above, the set Q is a subgroup of G of order 8. By constructing such subgroups from the combinations of τ_j's, γ_j's and ρ_j's, 15 subgroups of G of order 8, isomorphic to the Dihedral group D_8can be formed.

4.2 Subgroups of order 12

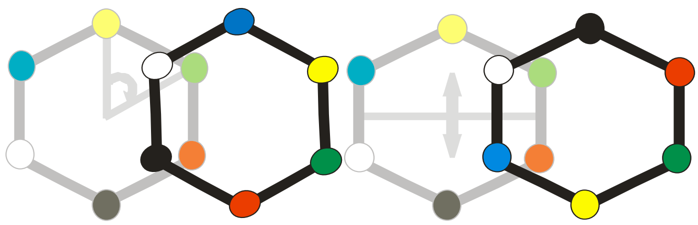
Since $12 = 2^2 * 3$, the direct product of S_2 and S_3 in S_5 is a subgroup of S_5. Hence, if T is a subgroup of S_5 of order 12, then it can be express as

$T = \{i, \rho_{10}, \alpha_1, \sigma_1, \sigma_2, \rho_5, \delta_1, \rho_9, \gamma_{15}, \gamma_3, \delta_2, \gamma_{14}\}$.

Hence, we can obtain 10 such subgroups of order 12.

Similarly, A_4 is obviously a subgroup of S_5, and each of the elements (1 2 3 4), (1 2 3 5), (1 2 4 5), (1 3 4 5) and (2 3 4 5) generate A_4. Thus, there are 5 such subgroups isomorphic to A_4.

Again, consider the regular 6-sided plane shape below.



a b

Figure 4.1: Regular 6-sided polygon

From O' level mathematics, the exterior angle of any regular n-gon is given by *$360^0/n$*. By rotating the regular n-gon through the angle *$360^0/n$* successively, we obtained all the set of rotations. Also, the flipping is taking along the lines of symmetries as shown in figure 1b, which is used to generate all the reflections.

We therefore generate the following:

- The identity operation leaving everything unchanged, id;
- Rotations of the n-gon by 60^0, 120^0, 180^0, 240^0, and 300^0 denoted by r_1, r_2, r_3, r_4 and r_5 respectively;
- Reflections about the vertical edge and the horizontal face (f_v and f_h) respectively, or through the four diagonals (f_{de}, f_{ce}, f_{df} and f_{cf}).

Hence, this is another group of order 12 given by $D_6 = \{id, r_1, r_2, r_3, r_4, r_5, f_v, f_h, f_{de}, f_{ce}, f_{df}, f_{cf}\}$.

The above elements are then used to generate the group table of D_6 below.

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•	***id***	r_1	r_2	r_3	r_4	r_5	f_v	f_{de}	f_{ce}	f_h	f_{df}	f_{cf}
id	id	r_1	r_2	r_3	r_4	r_5	f_v	f_{de}	f_{ce}	f_h	f_{df}	f_{cf}
r_1	r_1	r_2	r_3	r_4	r_5	id	f_{cf}	f_{df}	f_h	f_{de}	f_v	f_{ce}
r_2	r_2	r_3	r_4	r_5	id	r_1	f_{ce}	f_v	f_{de}	f_{df}	f_{cf}	f_h
r_3	r_3	r_4	r_5	id	r_1	r_2	f_h	f_{cf}	f_{df}	f_v	f_{ce}	f_{de}
r_4	r_4	r_5	id	r_1	r_2	r_3	f_{de}	f_{ce}	f_v	f_{cf}	f_h	f_{df}
r_5	r_5	Id	r_1	r_2	r_3	r_4	f_{df}	f_h	f_{cf}	f_{ce}	f_{de}	f_v
f_v	f_v	f_{df}	f_{de}	f_h	f_{ce}	f_{cf}	id	r_2	r_4	r_3	r_1	r_5
f_{de}	f_{de}	f_h	f_{ce}	f_{cf}	f_v	f_{df}	r_4	id	r_2	r_1	r_5	r_3
f_{ce}	f_{ce}	f_{cf}	f_v	f_{df}	f_{de}	f_h	r_2	r_4	id	r_5	r_3	r_1
f_h	f_h	f_{ce}	f_{cf}	f_v	f_{df}	f_{de}	r_3	r_5	r_1	id	r_4	r_2
f_{df}	f_{df}	f_{de}	f_h	f_{ce}	f_{cf}	f_v	r_5	r_1	r_3	r_2	id	r_4
f_{cf}	f_{cf}	f_v	f_{df}	f_{de}	f_h	f_{ce}	r_1	r_3	r_5	r_4	r_2	id

The defining operation of this group is function composition. Hence, the twelve symmetries are functions on the hexagon. The elements R = {id, r_1, r_2, r_3, r_4, r_5} form a subgroup of D_6, highlighted in green color (upper left region). A left and right cosets of this subgroup is also highlighted (in the last row, blue) and (last column, red), respectively.

Given this set of symmetries and the described operation, the group axioms are satisfied.

Performing the group operation such as $f_v \bullet r_3 = f_h$, i.e. flipping vertically after rotation by 180^0 equals flipping along the horizontal axis (f_h). Indeed every other combination of two symmetries still gives a symmetry, as can be checked using the group table. Hence, it is closed. Next, $(r_2 \bullet f_v) \bullet f_{cf} = r_2 \bullet (f_v \bullet f_{cf})$, i.e.

$(r_2 \bullet f_v) \bullet f_{cf} = f_{ce} \bullet f_{cf} = r_1$ and $r_2 \bullet (f_v \bullet f_{cf}) = r_2 \bullet r_5 = r_1$

This is through for all the elements of D_6 which shows the associativity. The identity element is the symmetry id leaving everything unchanged. Each of the transformations id, f_v, f_{de}, f_{ce}, f_h, f_{df}, f_{cf} and r_3 is its own inverse, because performing each one twice brings the hexagon back to its original orientation, id. The rotations r_1 and r_5, r_2 and r_4 are each other's inverse, i.e. $f_v \bullet f_v = id$, $r_1 \bullet r_5 = r_5 \bullet r_1 = id$.

In contrast to the group of integers, where the order of the operation is irrelevant, it does matter in D_6. That is, $f_v \bullet f_{de} = r_2$ but $f_{de} \bullet f_v = r_4$. In other words, D_6 is not abelian.

4.3 Developing the Concepts of Group and Subgroups Using GAP

In this section, we construct new groups from known groups using direct product of two or more groups. The fact that for any prime number $p > 2$ and $n \in Z^+$, $|U(p^n)| = |U(2p^n)|$ was established and also used symmetries to construct groups and their respective subgroups, characteristics and the unique factorization of the elements. Some functions f_i, $i \in Z^+$, on finite group G such that each f_i is a morphism were constructed. This follows from the fact that if G is Abelian and finite and H<G is a subgroup of G,

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then the factor group G/H is a finite Abelian group. The results also shows that if |G| = n and $n = r \cdot s \cdot t$, then $G \cong Z_r \otimes Z_s \otimes Z_t$ where $r,s,t \in Z^+$.

Now, let $U(n)$ be the set of all positive integers less than n and relatively prime to n. Then $U(n)$ is a group under multiplication modulo n. We shall begin with the help of GAP, by making a conjecture about the size of the group $U(pq)$ in terms of the groups $U(p)$ and $U(q)$ where p and q are relatively prime numbers greater than 2.

Let p = 11 and q = 13, then we obtained U(11), U(13) and U(143) using GAP as follows:

```
gap>ulist(11);
[ Z(11)^0, Z(11), Z(11)^8, Z(11)^2, Z(11)^4, Z(11)^9, Z(11)^7, Z(11)^3, Z(11)^6, Z(11)^5 ]
gap> Size(ulist(11));10
gap>ulist(13);
[ Z(13)^0, Z(13), Z(13)^4, Z(13)^2, Z(13)^9, Z(13)^5, Z(13)^11, Z(13)^3, Z(13)^8, Z(13)^10, Z(13)^7, Z(13)^6 ]
gap> Size(ulist(13));12

gap> Size(ulist(11))*Size(ulist(13));120
gap> Size(ulist(143));120
gap> (Size(ulist(143)))=(Size(ulist(11))*Size(ulist(13)));
true
```

From the above conjecture, we have seen that the order $|U(11)| \cdot |U(13)| = |U(143)| = 120$. Hence, $U(11) \otimes U(13) \cong U(143)$, where U(143) is the new group obtained from the product of U(11) and U(13).

The output ZmodnZObj(5, 143) for example, means the element 5 mod 143. We will now generate different subgroups for each group. For example in U(143), the cyclic subgroup generated by ZmodnZObj(5, 143) is:

```
gap> cyclic(143, 5);
[ZmodnZObj( 5, 143 ), ZmodnZObj( 25, 143 ), ZmodnZObj( 125, 143 ),
ZmodnZObj( 53, 143 ), ZmodnZObj( 122, 143 ),ZmodnZObj( 38, 143 ),
ZmodnZObj( 47, 143 ), ZmodnZObj( 92, 143 ), ZmodnZObj( 31, 143 ),
ZmodnZObj( 12, 143 ),ZmodnZObj( 60, 143 ), ZmodnZObj( 14, 143 ),
ZmodnZObj( 70, 143 ), ZmodnZObj( 64, 143 ), ZmodnZObj( 34, 143 ),
ZmodnZObj( 27, 143 ), ZmodnZObj( 135, 143 ), ZmodnZObj( 103, 143 ), ZmodnZObj( 86, 143 ), ZmodnZObj( 1, 143 )
]
gap> Size(cyclic(143, 5));
20
```

When different values of n, p and q as defined above were taken, gives more group structures and their respective subgroups.

The next conjecture is about the relationship between the size of the groups $U(p^k)$ and $U(2p^k)$ where p is a prime number greater than 2, and k is any positive integer. Now let p = 3 and k = 2.

```
gap>ulist(9);
[ ZmodnZObj( 1, 9 ), ZmodnZObj( 2, 9 ), ZmodnZObj(4,9), ZmodnZObj(5,9), ZmodnZObj( 7, 9 ), ZmodnZObj( 8, 9 ) ]
gap> Size(ulist(9));
6
gap>ulist(18);
[ ZmodnZObj( 1, 18 ), ZmodnZObj( 5, 18 ), ZmodnZObj(7,18), ZmodnZObj(11,18 ), ZmodnZObj( 13, 18 ),ZmodnZObj( 17, 18 ) ]
gap> Size(ulist(18));
6
gap> Size(ulist(9))=Size(ulist(18));
true
```

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The above result shows that $|U(p^k)| = |U(2p^k)|$. We therefore conclude that the two groups are isomorphic to each order. This is true for all prime numbers $p>2$. For $p = 2$, $|U(2p^k)| = 2|U(p^k)|$.

Again, consider the direct product of the cyclic subgroup C_8 of S_8 with the Symmetric group S_4. If we denote the direct product by D, then $D = C_8 \otimes S_4$ as presented below:

```
gap> C8:= CyclicGroup(IsPermGroup, 8);
Group([ (1,2,3,4,5,6,7,8) ])
gap> Size(C8);
8
gap> S4:= SymmetricGroup(4);
Sym( [ 1 .. 4 ] )
gap> Size(S4);
24
gap> D:= DirectProduct(C8, S4);
Group([ (1,2,3,4,5,6,7,8), (9,10,11,12), (9,10) ])
gap>orderFrequency(D);
[ [ 1, 1 ], [ 2, 19 ], [ 3, 8 ], [ 4, 44 ], [ 6, 8 ], [ 8, 64 ], [ 12, 16 ], [ 24, 32 ] ]
gap> Size(D);
192
gap> (Size(C8)*Size(S4))=Size(D);true
gap>IsNormal(D, C8);
true
gap>IsNormal(D, S4);
false
```

From the above results, the constructed group D is isomorphic to the direct product $C_8 \otimes S_4$. The subgroup C_8 of D is normal in D while the subgroup S_4 is not normal in D. The output order Frequency(D) means the group D has one element of order 1, nineteen elements of order 2, eight elements of order 3, fourty four elements of order 4, eight elements of order 6, sixty four elements of order 8, sixteen elements of order 12 and thirty two elements of order 24.

Next, we formulate some concrete groups based on the movements of the edges of a cube, take Rubik's cube as an example and label the eight vertices with numbers 1 to 8. We shall use G* to denote the group of the rotational symmetries of the cube (of order 8) which is a subgroup of the symmetric group S_8. Note that each rotation is 90^0, (e.g. r = (1, 2, 3, 4)(5, 6, 7, 8) is a rotation through 90^0) (Samaila, 2018).

```
gap> S:= SymmetricGroup(8);
Sym( [ 1 .. 8 ] )
gap> r:= (1, 2, 3, 4)(5, 6, 7, 8);;
gap> H:= Subgroup(S, [r]);
Group([ (1,2,3,4)(5,6,7,8) ])
gap> s:= (1, 5, 8, 4)(2, 6, 7, 3);;
gap> R:= Subgroup(S, [s]);
Group([ (1,5,8,4)(2,6,7,3) ])
gap> t:= (1, 2, 6, 5)(3, 7, 8, 4);;
gap> K:= Subgroup(S, [t]);
Group([ (1,2,6,5)(3,7,8,4) ])
gap> Size(H); Size(R); Size(K);
4
4
4
gap> H = R; H = K; R = K;
false
```

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Volume 8, Issue 9, pp 8-26, September-2024
https://doi.org/10.47001/IRJIET/2024.809002

```
false
false
gap> L:= Subgroup(S, [r, t]);
Group([ (1,2,3,4)(5,6,7,8), (1,2,6,5)(3,7,8,4) ])
gap> Size(L);
24
gap>IsCyclic(L);
false
gap> u:= (1,2,4,5,8,6,7,3);;
gap> v:= (2,4,6,8);;
gap> M:= Subgroup(S, [u, v]);
Group([ (1,2,4,5,8,6,7,3), (2,4,6,8) ])
gap> Size(M);
40320
gap>IsCyclic(M);
false
gap>IsNormal(S, M);
true
gap> S = M;
true
gap> Factorization(M, (1,8,3,6,4,5,2,7));
x2^-1*x1^2*x2^2
gap> Factorization(M, (1,6,4,5,3,2,7,8));
x2^2*x1^-1*x2^2*x1^2*x2^-1*x1
gap> Factorization(M, ((1,3,5,7)(2,4,6,8)));
x2^2*x1^-1*(x2^-1*x1^2)^2*x1
gap> Factorization(M, ((1,8)(2,7,4)(3,6,5)));
x1*x2^-1*x1^-2*x2*x1*x2*x1^-2
gap> Factorization(M, ((1,4,2)(3,5,6,8,7)));
x1^-1*x2^-1*(x1^2*x2)^2*x1^-1*x2*x1^2
gap> Factorization(M, (3,8));
x2*x1^-1*(x2*x1)^2*x2^-1*x1^4
gap> quit;
```

It is clear that every rotation of the cube is in the subgroup L. Thus G* = L and hence, G* ≅ L. Also, the subgroups H, R and K of G* are distinct proper subgroups of G*. Again, the output Factorization(M, (1,8,3,6,4,5,2,7)) = x2^-1*x1^2*x2^2 tells us that (1,8,3,6,4,5,2,7) = $(2,4,6,8)^{-1}$ * $(1,2,4,5,8,6,7,3)^{2}$ * $(2,4,6,8)^{2}$ where x1 and x2 are the first and the second generators of the group M respectively, with M = S.

Finally, let *f* be a function mapping a group G onto itself, where G is a cyclic subgroup of the permutation group S_8 as follows:

```
gap> G:= CyclicGroup(IsPermGroup, 8);
Group([ (1,2,3,4,5,6,7,8) ])
gap> Elements(G);
[ (), (1,2,3,4,5,6,7,8), (1,3,5,7)(2,4,6,8), (1,4,7,2,5,8,3,6), (1,5)(2,6)(3,7)(4,8), (1,6,3,8,5,2,7,4), (1,7,5,3)(2,8,6,4),
(1,8,7,6,5,4,3,2) ]
gap> r:= G.1;
(1,2,3,4,5,6,7,8)
gap> f:= x -> x^5;
function( x ) ... end
gap> N:= Subgroup(G, [f(r)]);
Group([ (1,6,3,8,5,2,7,4) ])
```

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```
gap> Elements(N);
[ (), (1,2,3,4,5,6,7,8), (1,3,5,7)(2,4,6,8), (1,4,7,2,5,8,3,6), (1,5)(2,6)(3,7)(4,8), (1,6,3,8,5,2,7,4), (1,7,5,3)(2,8,6,4), (1,8,7,6,5,4,3,2) ]
gap> Size(N);
8
gap> Size(G) = Size(N);
true
gap> N = G;
true
gap> f:= x -> x^4;
function( x ) ... end
gap> M:= Subgroup(G, [f(r)]);
Group(())
gap> Elements(M);
[ () ]
gap> Size(M);
1
gap> f:= x -> x^6;
function( x ) ... end
gap> K:= Subgroup(G, [f(r)]);
Group([ (1,7,5,3)(2,8,6,4) ])
gap> Elements(K);
[ (), (1,3,5,7)(2,4,6,8), (1,5)(2,6)(3,7)(4,8), (1,7,5,3)(2,8,6,4) ]
gap> Size(K);
4
gap> Size(G)/Size(K);
2
```

The subgroup N of G is the image of G under the function $f(x) = x^5$. The order of the subgroup N is 8, equal to the order of G and the result shows that N = G. Hence the function *f* is an *automorphism*. But the images M and K of G under the functions $f(x) = x^4$ and $f(x) = x^6$ respectively, are proper subgroups of G, where M is the trivial subgroup of G. The pre-image of M under the function $f(x) = x^4$ gives the kernel of the function. Also, the index [G **:** K] of the subgroup K in G is 2. Hence, the subgroup N is normal in G, i.e. $N \lhd G$.

4.4 Generating Group Actions for Network Dynamics and Groupoid Formalism

Let $(X, \pi) = \{\phi_1, \phi_2, ..., \phi_n\}$ be a Space where π is the representative of the functions over *X* and let $\sigma \in S_n$ with $n \geq 2$. Then the group $S_n$acts on the space *X* by permuting its elements as follows:

$$(\sigma \cdot \pi)(\phi_1, \phi_2, ..., \phi_n) = \pi(\phi_{\sigma(1)}, \phi_{\sigma(2)}, ..., \phi_{\sigma(n)}) \quad 4.1$$

Every element $\sigma \in S_n$ satisfy $\phi_i \mapsto \phi_{\sigma(i)}$ in $\pi(\phi_1, \phi_2, ..., \phi_n)$.

Lemma 4.4.1: The function defined in Equation 4.1 is a group action of S_n on the signal space *X*.

Proof: Obviously, $i \cdot \pi = \pi$. Next, we show that $\sigma \cdot (\delta \cdot \pi) = (\sigma\delta) \cdot \pi$ for all $\sigma, \delta \in S_n$. Now,

$$\begin{aligned}(\sigma \cdot (\delta \cdot \pi))(\phi_1, \phi_2, ..., \phi_n) &= (\delta \cdot \pi)(\phi_{\sigma(1)}, \phi_{\sigma(2)}, ..., \phi_{\sigma(n)}) \\ &= \pi(\phi_{\sigma(\delta(1))}, \phi_{\sigma(\delta(2))}, ..., \phi_{\sigma(\delta(n))}) \\ &= \pi(\phi_{(\sigma\delta)(1)}, \phi_{(\sigma\delta)(2)}, ..., \phi_{(\sigma\delta)(n)})\end{aligned}$$

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$$= ((\sigma\delta)\cdot\pi)(\phi_{(1)},\phi_{(2)},...,\phi_{(n)}).$$

Hence, the result follows.

4.4.1 Generating Transfer Functions

We shall construct a new homomorphism called transfer function, which is a mathematical function that gives possible output values for every corresponding input values. Now, a Transversal of a subgroup in a group is defined as follows: let G be a group and H be a non-trivial subgroup of G such that $[G:H] = r$ for some $r\in Z^+$ and $r\leq n$. Then a collection T which contain exactly one representative from each coset (left or right) of H in G is called the transversal of H in G. Note that if $T = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ and T is a transversal of H in G, then

$$G = \bigcup_{i=1}^{n} \alpha_i H .$$

Also, $\alpha_i H \bigcap \alpha_j H = \Phi$ for $i\neq j$ and whenever $\beta \in \alpha H$, then $\beta H = \alpha H$. Again, given any finite set X and a bijection $\zeta: X\to X$, a collection of all such bijections on X is called a symmetric group on X denoted by S_n of order $n!$ Thus if $\sigma\in S_n$, then σ is considered as a transformation on S_n.

Below are some facts about transversals of a subgroup in a group.

Lemma 4.4.2: Let H be a non-trivial Abelian subgroup of a group G such that $[G:H] = r$. If $T = \{\alpha_1, \alpha_2, ..., \alpha_n\}$ is a transversal of H in G and $g\in G$, then

i. there exists $h_r \in H$ and $\delta \in G$ depending on g such that $g\alpha_r = \alpha_{r\delta} h_r$;

ii. if $\{\beta_1, \beta_2, ..., \beta_n\}$ is another transversal of H in G and $k_r \in H$, $\delta \in G$, then

$$\prod_{r=1}^{n} h_r = \prod_{r=1}^{n} k_r .$$

Proof: (i) Since $\alpha_r \in H$, $g\alpha_r \in xH$ for some x. Thus, we can find $h_r \in H$ and for a positive integer r, there exists q_r such that $g\alpha_r = \alpha_{q_r} h_r$ with $r\leq n$. Now, to show that q_r defines a transformation in G, it is sufficient to show that q_r is injective.

Suppose $g\alpha_{r'} = \alpha_{q_{r'}} h_{r'}$ for some $h_{r'} \in H$ and $q_r = q_{r'}$, then we have

$$\alpha_r^{-1}\alpha_{r'} = (g\alpha_r)^{-1}(g\alpha_{r'}) = (\alpha_{q_r} h_r)^{-1}(\alpha_{q_{r'}} h_{r'}) = h_r^{-1} h_{r'} \in H .$$

This implies that

$$\alpha_r H = \alpha_{r'} H \text{ (since } h_r^{-1} \in H,\ h_{r'}^{-1} \in H\text{)}.$$

Hence, $r = r'$ since T is a transversal. This shows that the transformation q is injective and thus, a bijection (since it is an injection of a finite set to itself) of n-elements. Thus, $q \in G$.

(ii) Suppose now that $q = \delta \in G$. Then from (i) with $g = e$, there exists $p_r \in H$ and $\xi \in G$ such that $\beta_r = \alpha_{r\xi} p_r$ for $1 \leq r \leq n$ and using the identity $\alpha_{r\xi\delta} = \alpha_{r\xi\delta\xi\xi^{-1}}$, we have

$$g\beta_r = g(\alpha_{r\xi} p_r) = (\alpha_{r\xi\delta} h_{r\xi}) p_r = \beta_{r\xi\delta\xi^{-1}} (p_{r\xi\delta\xi^{-1}})^{-1} h_{r\xi} p_r .$$

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Volume 8, Issue 9, pp 8-26, September-2024
https://doi.org/10.47001/IRJIET/2024.809002

Then since $\xi\delta\xi^{-1} \in G$ and $(p_{r\xi\delta\xi^{-1}})^{-1} h_{r\xi} p_r \in H$, set $k = \xi\delta\xi^{-1}$ and $f_r = (p_{r\xi\delta\xi^{-1}})^{-1} h_{r\xi} p_r$. Thus, $g\beta_r = \beta_{rk} f_r$ as required and since H is Abelian,

$$\prod_r f_r = \prod_r (p_{r\xi\delta\xi^{-1}})^{-1} h_{r\xi} p_r = \prod_r p_r^{-1} \prod_r h_r \prod_r p_r = \prod_r h_r$$

Since $\xi, \delta \in G$ for $r = 1, 2, \ldots, n$. Hence, the result follows.

Now, from the construction above, we defined a transfer as a homomorphism from G to H as follows.

Definition 4.4.3: Let G be a group (not necessarily Abelian) and let H be an Abelian subgroup of G such that $[G{:}H] = r$. With the notation set up in Lemma 4.3.1, the function $\xi : G \to H$ defined by $g\xi = \prod_{r=1}^{n} h_r$ for all $g \in G$ is called the transfer of H in G.

But this is independent of the coset representatives h_r of H in G, thus, the function is well-defined. Next we shall show that the transfer ξ is a homomorphism.

Theorem 4.4.4: Let G be a group and H be a subgroup of G such that $[G{:}H] = r$. Then the transfer of H in G, given by $\xi : G \to H$ is a homomorphism.

Proof: Let $g, q \in G$, $g\alpha_r = \alpha_{r\delta} p_r$ and $q\alpha_r = \alpha_{r\sigma} k_r$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a transversal. Again, with $\delta, \sigma \in G$,

$$gq\alpha_r = g\alpha_{r\sigma} k_r = \alpha_{r\sigma\delta} p_{r\sigma} k_r .$$

Thus, since H is Abelian, we have

$$gq\xi = \prod_r p_{r\sigma} k_r = \prod_r p_r \cdot \prod_r k_r = g\xi \cdot q\xi .$$

Hence, ξ is a homomorphism.

4.4.2 Distributors

Distributors' measures the extent to which a transfer function ξ distributes over elements of a group with respect to multiplication. It is also important most especially in these days of electronic ignition where the concept is required in recycling. Let $\xi : G \to H$ be a transfer function between finite groups. Then we defined the ξ-distributor $[\sigma, \rho : \xi]$ for $\sigma, \rho \in G$ as

$$[\sigma, \rho : \xi] = \xi(\rho)^{-1} \xi(\sigma)^{-1} \xi(\sigma\rho) = \xi(\rho)^{-1} \xi^{\sigma}(\rho)$$

So that $\xi(\sigma\rho) = \xi(\sigma)\xi(\rho)[\sigma, \rho : \xi]$. The set of distributors in G also measures the extent to which the function ξ fails to be a homomorphism. We now give some properties of distributor ξ as follows:

Theorem 4.4.5: Suppose $\xi : G \to H$ is a transfer function from G to H and let $\sigma, \rho, \omega \in G$. Then the ξ-distributor satisfy

$$[\rho, \omega : \xi][\sigma, \rho\omega : \xi] = [\sigma, \rho : \xi]^{\xi(\omega)} [\sigma\rho, \omega : \xi].$$

Proof: The proof is a straight forward exercise when we expand $\xi(\sigma\rho\omega)$ in two different ways.

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Volume 8, Issue 9, pp 8-26, September-2024
https://doi.org/10.47001/IRJIET/2024.809002

$$\xi(\sigma\rho\omega) = \xi(\sigma)\xi(\rho\omega)[\sigma,\rho\omega:\xi] = \xi(\sigma)\xi(\rho)\xi(\omega)[\rho,\omega:\xi][\sigma,\rho\omega:\xi]$$

and

$$\xi(\sigma\rho\omega) = \xi(\sigma\rho)\xi(\omega)[\sigma\rho,\omega:\xi] = \xi(\sigma)\xi(\rho)[\sigma,\rho:\xi]\xi(\omega)[\sigma\rho,\omega:\xi].$$

Hence, the result follows.

Theorem 4.4.6: Suppose $\xi: G \to H$ is a transfer function from G to H and let $\sigma,\rho,\omega \in G$. Then the ξ-distributor satisfy

$$[\sigma\rho,\omega:\xi] = [\sigma,\omega:\xi][\rho,\omega:\xi^{\sigma}].$$

Proof: The proof is also straight forward by expanding both sides separately.

$$[\sigma\rho,\omega:\xi] = \xi(\omega)^{-1}\xi(\sigma\rho)^{-1}\xi(\sigma\rho\omega)$$

and

$$[\sigma,\omega:\xi][\rho,\omega:\xi^{\sigma}]. = \xi(\omega)^{-1}\xi(\sigma)^{-1}\xi(\sigma\omega)\xi^{\sigma}(\omega)^{-1}\xi^{\sigma}(\rho)^{-1}\xi^{\sigma}(\rho\omega)$$

$$= \xi(\omega)^{-1}\xi(\sigma)^{-1}\xi(\sigma\omega)(\xi(\sigma)^{-1}\xi(\sigma\omega))^{-1}(\xi(\sigma)^{-1}\xi(\sigma\rho))^{-1}(\xi(\sigma)^{-1}\xi(\sigma\rho\omega))$$

$$= \xi(\omega)^{-1}\xi(\sigma\rho)^{-1}\xi(\sigma\rho\omega).$$

This established the result.

Remark: 4.4.7:

Note that if $\tau \in G$, then the distributor of τ defines an operator ξ_{τ} on the function $\xi: G \to H$ such that $\xi_{\tau}(x) = [x,\tau:\xi]$. Theorem 4.3.9 therefore implies that $(\xi_{\tau}(x))^{g} = \xi_{\tau}^{g}(x)$ for all $g \in G$. Hence, the distributors operators commute with action of conjugation on functions.

Now, the elements of the group G act on the signal space X as functions. Hence in general, if X is any signal space and G is any subgroup of S_X, then X is a G-set under the group action

$$(\sigma,\phi(t)) \mapsto \sigma(\phi(t))$$

for all $\sigma \in$G and $\phi(t) \in$X.

Lemma 4.4.8: Let |G| = n such that G≅X. Then G acts on X by the left regular representation given by

$$(\sigma,\phi(t)) \mapsto \pi_{\sigma}(\phi(t)) = \sigma\phi(t).$$

Proof: Since π_{σ} is a left multiplication,

$$i\cdot\phi(t) = \pi_i\phi(t) = i\phi(t) = \phi(t)$$

where*i* is the identity element of G. Also,

$$(\sigma\delta)\cdot\phi(t) = \pi_{\sigma\delta}\phi(t) = \pi_{\sigma}\pi_{\delta}\phi(t) = \pi_{\sigma}(\delta\phi(t)) = \sigma\cdot(\delta\cdot\phi(t)).$$

This established the result.

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Lemma 4.4.9: Let $|G| = n$ such that $G \cong X$ and let K be a subgroup of G. Then X is a K-set under conjugation. i.e., an action $K \times X \to X$ of K on X defined by

$$(\delta, \phi(t)) \mapsto \delta(\phi(t))\delta^{-1}$$

for all $\delta \in K$ and $\phi(t) \in X$.

Proof: Clearly, the first axiom for group action is satisfied. Next, observed that

$$(\sigma\tau, \phi(t)) = \sigma\tau(\phi(t))(\sigma\tau)^{-1}$$

$$= \sigma\tau(\phi(t))(\tau^{-1}\sigma^{-1})$$

$$= \sigma(\tau(\phi(t))\tau^{-1})\sigma^{-1}$$

$$= (\sigma, (\tau, \phi(t)))$$

Which shows that the second condition is also satisfied as required.

Example 4.4.10: Suppose $G = S_n$ and choose $n = 5$. Then $G = S_5$ and $X = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5\} \in \Re^5$.

Now, from

$$(\sigma \cdot (\delta \cdot \pi))(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) = (\delta \cdot \pi)(\phi_{\sigma(1)}, \phi_{\sigma(2)}, \phi_{\sigma(3)}, \phi_{\sigma(4)}, \phi_{\sigma(5)}),$$

we have

$$(\sigma_3 \cdot (\delta_1 \cdot \pi))(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) = (\delta_1 \cdot \pi)(\phi_{(4)}, \phi_{(1)}, \phi_{(3)}, \phi_{(2)}, \phi_{(5)})$$
$$= (\phi_{(3)}, \phi_{(4)}, \phi_{(1)}, \phi_{(5)}, \phi_{(2)}).$$

But $\sigma_3 \cdot \delta_1 = \delta_{18}$ in G and

$$(\delta_{18} \cdot \pi)(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) = (\phi_{(3)}, \phi_{(4)}, \phi_{(1)}, \phi_{(5)}, \phi_{(2)}).$$

Hence, $(\sigma_3 \cdot (\delta_1 \cdot \pi))(w) = ((\sigma_3\delta_1) \cdot \pi))(w) = (\delta_{18} \cdot \pi)(w)$.

Example 4.4.11: Network signals are regarded as functions on some discrete groups, usually identified with the group of integers Z, or its subgroups Z_p of integer's modulo p, i.e. $\varphi : Z_p \to Z_p$. Let $p = 2$. Then Z_p is isomorphic to S_2. Now, a binary symmetric channel is described as a model consisting of a transmitter which is capable of sending a binary signal together with a receiver. Then one possible coding scheme is to send a signal several times so as to compare the received signals with one another. Suppose that the signal to be encoded is (1 1 0 1 0 0) into a binary 4n-tuple, let $\sigma \in S_n$. Then $\sigma : Z_2 \to Z_2$ encode (1 1 0 1 0 0) into a binary 4n-tuple as

$$(110100) \mapsto (110100110100110100110100)$$

The decoded signal depends on the function $\sigma \in S_n$. The function σ is also required to be one-to-one in order that two signals will not be encoded into the same image.

NOTE: Consider a transformation $T : G \to G;\ \omega \mapsto T(\omega)$ for some $\omega \in G$. Then we can write $T_\tau(\omega) = (\omega)^\tau = \tau(\omega) = (\omega)\tau^{-1}$ for all $\tau \in G$. This operation modify a signal in such a way that the signal is defined on the domain G and any transformation $\tau \in G$ is a mapping from G to G. Thus, the transformed signals are given by $\omega^\tau(x(n)) = \omega\tau^{-1}(x(n))$ where $x(n)$ is some discrete signal.

Finally, agroupoid is a small Category in which every morphism is invertible.The standard definition of a category C involves a collection of objects, A, B,..., and a set of morphisms (which could be the empty set) for each pair of objects; Hom(A, B) for

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objects A and B. The composition of morphisms is defined and is associative, and there is an identity element in each Hom(A, A), therefore Hom(A, A) is never empty. But a category C can be viewed as an algebraic structure in itself, endowed with a binary operation, making it similar to a group or semigroup. We call this associated algebraic structure G(C). The "elements" of G(C) are the morphismsof C, and the "product" is the composition, which is an associative partial binary operation with identity elements. If C has only one object, then any two morphisms can be composed, and we have only one identity element. The axioms of a category guarantee that G(C) is a semigroup. Furthermore, if we insist on the invertibility of each morphism in C, then G(C) is a group.In the groupoid approach, we examined not the symmetry of the small subnetworks and motifs, but rather the dynamics of these small networks, when they are directed graphs, and in particular when these small nets are wired together to make larger networks (circuits).

V. CONCLUSION

This paper clearly presented the concept of group theory in more simplified version especially the methods of constructing and generating subgroups of a group of different order. We have successfully developed some methods for constructing groups of symmetries, which can be used for concrete applications in other field of sciences. We have shown that the transversal of subgroups in a group behaves the same way with cosets decomposition where $\alpha_i H \cap \alpha_j H = \phi$ for all $i \neq j$ and if $\alpha_i H \cap \alpha_j H \neq \phi$, then $\alpha_i H = \alpha_j H$. The result also proved that the transfer function $\xi : G \to H$ is a homomorphism and if $H = G$, then ξ is an isomorphism that decomposed one transformation into multiple transformations and then distributes over elements of G with respect to multiplication. The results also shows that the representations of S_n act on the signal space (X,π) by transforming $\phi_i \to \phi_{\sigma(i)}$ for all $\phi \in (X,\pi)$ and $\sigma \in S_n$,where each $\sigma \in S_n$ can be used to transform signal from one form to another in a manner that each representation act on a signal as input, processed it using permutation and then produced the result as output. The generated subgroups can be used on representations of the cyclic group H to produce network signals that are spiral (Signals that are bounded by elements of cyclic group). The path of such signal is the same with the signal presented below (Figure 2).

Since signals are regarded as functions on groups into fields and Discrete signals as signals on some discrete groups usually identified with the group of integers Z, or with some of its subgroups Z_p of integer's modulo p, we defined a function $\xi : Z \to X$ by $\xi(n) = \rho^{2k\pi/n}$ for all $n \in Z^+$, $k \geq 0$ where X could be the complex field C, the real field R, or some finite fields. Then ξ distribute every $n \in$ Z over X as follows:

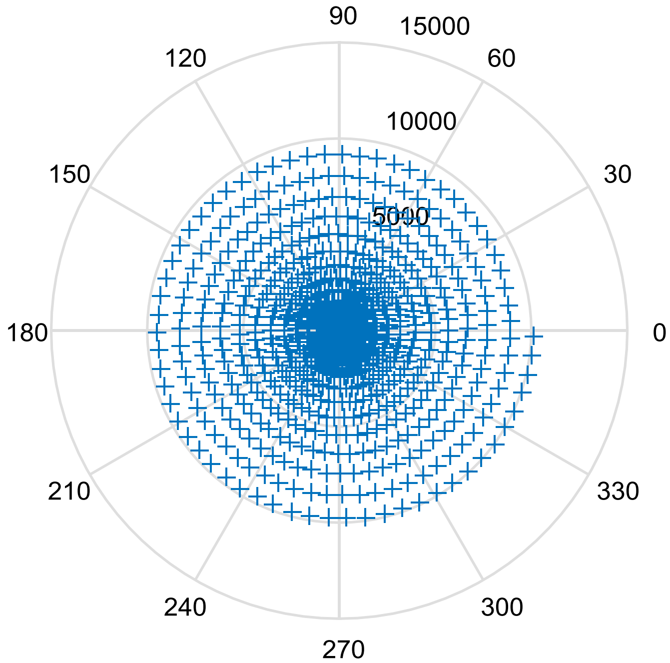

Figure 5.1: Path of signal generated by cyclic groups

A transformation $\xi : G \to G$ in this case, defined by $\tau \mapsto \xi(\tau)$ for all $\tau \in G$ decomposed an element $\tau \in G$ into the product $\eta\varphi\gamma \in G$ over a discrete signal $x(n)$ given by $\tau(x(n)) = \eta(\varphi(\gamma(x(n))))$.

It is therefore expected that the problem of understanding abstract Mathematics will be addressed at the end of this research. Students should be able to apply it in other field of sciences such as Cryptography, Signal processing, Image processing, Coding and Crystallography. This research will definitely explore clearly the level of students' knowledge and understanding of group theory in Mathematics and how an individual can develop a better understanding of some topics in the given domain.

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