On Certain Types of C^{hv} - Mixed Trirecurrent Finsler Space

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Abstract - **In Finsler geometry, there are two types of Cartan's covariant derivatives that are called $v -$covariant derivative and $h -$covariant derivative. In this paper, we deal with both derivatives simultaneously and introduce a $C^{hv} -$mixed trirecurrent space. Furthermore, we study special spaces, such as $P2 -$ Like space and $P^* -$ space in the main space.**

Keywords: C^{hv}–mixed trirecurrent space, $v -$covariant derivative, $h -$covariant derivative, $P2 -$ Like $-$space and $P^* -$ space.

I. Introduction and Preliminaries

Understanding the relationship between Cartan's covariant derivatives and curvature tensors is essential for the study of special Finsler spaces, such as recurrent, symmetric, and generalized curvature-restricted spaces. In particular, many geometric conditions and classification results in Finsler geometry are formulated in terms of the behavior of curvature tensors under $h -$ and $v -$covariant differentiation.

The concept of $C^h -$recurrent space and $C^v -$birecurrent space are introduced by Matsumoto, Mishra and Lodhi [11, 12]. Al-Qashbari et al. [3, 4] discussed the generalized birecurrent Finsler space of mixed covariant derivatives in Cartan sense and generalized trirecurrent Finsler space under Cartantype mixed covariant derivatives. Additionally, the properties of $C^{hv} -$mixed birecurrent Finsler spaces within the Cartan sense have been studied by Al- Qashbari et al.[2].

The $P2 -$ Like $-$space and $P^* -$space in generalized recurrent Finsler spaces of order one, two, three have been studied by [1, 6, 9]. The vectors y_i and y^i defined by

$$(1.1) \quad a) \ y_i = g_{ij}(x, y)y^j \text{ and } \qquad b) \ \dot{\partial}_j \ y_i = g_{ij}$$

The metric tensor g_{ij} and its associative g^{ij} are connected by [5, 10]

$$(1.2) \quad g_{ij} \ g^{ik} = \delta_j^k = \begin{cases} 1 \ if \ j = k \ , \\ 0 \ if \ j \neq k \ . \end{cases}$$

In view of (1.1) and (1.2), we have

$$(1.3) \quad a) \ \delta_j^i y_i = y_j \ , \ b) \ \delta_j^i \ y^j = y^i, c) \ \delta_j^i \ g_{ir} = g_{jr}, \ d) \ \delta_j^i \ g^{jk} = \ g^{ik},$$

$$e) \ y_i \ y^i = F^2 \ , \qquad f) \ \delta_i^i = n \text{ and } \qquad g) \ \dot{\partial}_j y^i = \delta_j^i \ .$$

The $(h)hv -$torsion tensor which is positively homogeneous of degree -1 in y^i and symmetric in all its indices introduced and defined by [11]

$$(1.4) C_{ijk} = \frac{1}{2}\dot{\partial}_i \ g_{jk} = \frac{1}{4}\dot{\partial}_i \ \dot{\partial}_j \ \dot{\partial}_k \ F^2 \ .$$

And satisfies

$$(1.5) \ a) \ C_{ijk} \ y^i = C_{kij} \ y^i = C_{jki} \ y^i = 0, b) \ C_{jk}^h \ g_{ih} = C_{ijk} \ , \quad c) \ C_{jk}^h \ y^j = 0 = C_{jk}^h y^k$$

$$d) \ C_{jk}^h \ g^{jk} = C^h, e) \ C_{ijk} \ g^{jk} = C_i \ , f) \ C_{jh}^h = C_j, \quad g) \left(\dot{\partial}_i C_{jk}^h\right) y^k = -C_{ji}^h,$$

h) $C_i\, y^i = 0$ and i) $C_{ijk}\, g^{ih} = C_{jk}^h$,

where C_{jk}^h is $(v)hv-$ torsion tensor.

Cartan [7] deduced the covariant derivatives of an arbitrary vector field X^i with respect to x^k which given by

$$(1.6) \quad X^i\big|_k = \dot{\partial}_k X^i + X^r C_{rk}^{\,i}$$

and

$$(1.7) \quad X_{|k}^i = \partial_k X^i - \left(\dot{\partial}_r X^i\right) G_{\,k}^{\,r} + X^r \Gamma_{rk}^{*\,i} ,$$

where the function $\Gamma_{rk}^{*\,i}$ is defined by $\Gamma_{rk}^{*\,i} = \Gamma_{rk}^{\,i} - C_{mr}^{\,i}\, \Gamma_{sk}^{\,m}\, y^s$.

$X^i|_k$ and $X_{|k}^i$ are $v-$covariant derivative and $h-$covariant derivative of the vector field X^i. Therefore, $v-$covariant derivative and $h-$covariant derivative of the vectors y^i , y_i and metric tensors g_{ij} and its associative g^{ij} are satisfied [10]

$$(1.8) \quad \text{a)}\; g_{ij}\big|_k = 0, \quad \text{b)}\; g_{ij\,|k} = 0 , \quad \text{c)}\; g^{ij}\big|_k = 0 , \quad \text{d)}\; g_{|k}^{ij} = 0 ,$$

$$\text{e)}\; y_{|k}^i = 0 , \qquad \text{f)}\; y^i\big|_k = \delta_k^i , \qquad \text{g)}\; y_{j\,|k} = 0 \;\text{ and }\; \text{h)}\; y_j\big|_k = g_{jk} .$$

The curvature tensor $P_{jk\,h}^i$ is positively homogeneous of degree zero in y^i and the tensors $P_{jk\,h}^i$, associative tensor P_{rkh} and curvature vector P_k satisfies

$$(1.9) \quad \text{a)}\; P_{jk\,h}^i y^j = \Gamma_{jk\,h}^{*i} y^j = P_{kh}^i = C_{kh|r}^i y^r ,\text{b)}\; P_{ijk\,h} = g_{ir} P_{jk\,h}^r, \quad \text{c)}\; P_{jki}^i = P_{jk}$$

$$\text{d)}\; P_{ki}^i = P_k , \; \text{e)}\; P_{kh}^i g_{ir} = P_{rkh}\text{and} \quad \text{f)}\; (\dot{\partial}_k \Gamma_{jh}^{*i}) y^j = P_{kh}^i = (\dot{\partial}_k \Gamma_{hj}^{*i}) y^j .$$

The tensor $S_{jk\,h}^i$ called Cartan's first curvature tensorthat defined by

$$(1.10) \quad S_{jk\,h}^i = C_{kr}^i C_{jh}^r - C_{rh}^i C_{jk}^r \; .$$

The associate curvature tensor $S_{jpk\,h}$, Ricci tensor S_{jk}, deviation tensor S_j^i and curvature scalar S of the curvature tensor $S_{jk\,h}^i$ are given by

$$(1.11)\; \text{a)}\; S_{jpk\,h} = g_{ip} S_{jk\,h}^i , \quad \text{b)}\; S_{jki}^i = S_{jk} , \quad \text{c)}\; S_j^i = S_{jk} g^{ik} \;\text{ and }\; \text{d)}\; S_{jk}\, g^{jk} = S .$$

Since the $(h)hv-$ torsion tensor C_{jk}^i defined by [2]

$$(1.12) \quad C_{jk\,|l}^i = \lambda_l C_{jk}^i + \mu_l(\delta_k^i y_j - \delta_j^i y_k), \qquad C_{jk}^i \neq 0,$$

where λ_l and μ_l are non -zero recurrent covariant vectors field.

Transvecting (1.12) by g_{ih} ,using (1.5b), (1.3c) and (1.8b), we get

$$(1.13) \quad C_{jhk|l} = \lambda_l C_{jhk} + \mu_l(g_{kh} y_j - g_{jh} y_k).$$

Now, taking $h-$ covariant derivative for (1.12), with respect to x^m, we get

$$C_{jk\,|l|m}^i = \lambda_{l|m} C_{jk}^i + \lambda_l C_{jk\,|m}^i + \mu_{l|m}(\delta_k^i y_j - \delta_j^i y_k).$$

Using the condition (1.12) in the above equation, we get

$$C_{jk\,|l|m}^i = \lambda_{l|m} C_{jk}^i + \lambda_l\big[\lambda_m C_{jk}^i + \mu_m(\delta_k^i y_j - \delta_j^i y_k)\big] + \mu_{l|m}(\delta_k^i y_j - \delta_j^i y_k).$$

Or

$$(1.14) \quad C^i_{jk\,|l|m} = e_{lm} C^i_{jk} + f_{lm}(\delta^i_k y_j - \delta^i_j y_k), \qquad C^i_{jk} \neq 0$$

where $|l|m$ denoted the $h-$covariant differentiation of second order, where $e_{lm} = \lambda_{l|m} + \lambda_l \lambda_m$ and $f_{lm} = \lambda_l \mu_m + \mu_{l|m}$ are non-zero covariant tensors field of second order .

Remark 1.1.A Finsler space F_n which $(h)hv-$ torsion tensor C^i_{jk} satisfies the condition (1.14) called a C^h-generalized birecurrentFinsler space of second order and denoted it by $C^h - GBRF_n$.

Transvecting the condition (1.14) by g_{ih} ,using (1.5b),(1.3c) and (1.8b), we get

$$(1.15) \quad C_{jhk|l|m} = e_{lm} C_{jhk} + f_{ml}(g_{kh} y_j - g_{jh} y_k).$$

Contracting the indices i and k in the condition (1.14), using (1.5c), (1.3a) and (1.3f), we get

$$(1.16) \quad C_{j\,|l|m} = e_{lm} C_j + f_{ml}(n-1)y_j .$$

Transvecting the condition (1.14) by g^{jk} , using (1.5d),(1.3b) and (1.8d),we get

$$(1.17) \quad C^i_{|l|m} = e_{lm} C^i .$$

AL- Qashbari et al. [2] introduced the $C^{hv}-$mixed birecurrence Finsler space, i.e. characterized by the condition

$$(1.18) \quad C^i_{jk\,|l|\,m} = a_{lm} C^i_{jk} + b_{lm}(\delta^i_k y_j - \delta^i_j y_k) + \mu_l(\delta^i_k g_{jm} - \delta^i_j g_{km}). C^i_{jk} \neq 0,$$

where $|l$ denote the $h-$covariant differentiation and $|m$ denote the $v-$ covariant differentiation, where $a_{lm} = \lambda_{l|m} + \lambda_l a_m$ and$b_{lm} = \mu_{l|m}$ are non$-$ zero covariant tensorsfield of second order and μ_l is non$-ze$ro covariant vector field of first order.

Also, the $(h)hv-$ torsion tensor $C_{jk\,h}$ in the same space of condition (1.18) given by [2]

$$(1.19) \quad C_{jhk|l|\,m} = a_{lm} C_{jhk} + b_{lm}(g_{kh} y_j - g_{jh} y_k) + \mu_l(g_{kh} g_{jm} - g_{jh} g_{km}).$$

II. On $C^{hv} -$ Mixed Trirecurrent space

Misra and Lodhi defined the C^v- recurrent Finsler space by the condition

$$(2.1) \quad C^i_{jk\,|n} = \lambda_n C^i_{jk} . \qquad C^i_{jk} \neq 0$$

Now, taking $v-$ covariant derivative for (1.14) with respect to x^n, we get

$$C^i_{jk\,|l|m\,|n} = e_{lm\,|n} C^i_{jk} + e_{lm} C^i_{jk\,|n} + f_{lm\,|n}(\delta^i_k y_j - \delta^i_j y_k) + f_{lm}(\delta^i_k y_j - \delta^i_j y_k)_{|n}.$$

Using the equation (2.1), (1.8h) in the above equation, we get

$$(2.2) \quad C^i_{jk\,|l|m\,|n} = c_{lmn} C^i_{jk} + d_{lmn}(\delta^i_k y_j - \delta^i_j y_k) + f_{lm}(\delta^i_k g_{jn} - \delta^i_j g_{kn}),$$

where $|l|m$ denoted the $h-$ covariant differentiation of second order and $|n$ denote the $v-$ covariant differentiation of first order,where $c_{lmn} = e_{lm\,|n} + e_{lm} \lambda_n$ and $d_{lmn} = f_{lm\,|n}$are non -zero covariant tensors field of third order and b_{lm} is non -zero covariant tensor field of second order.

Definition 2.1: The space which the $(h)hv-$torsion tensor C^i_{jk} satisfies the condition (2.2) will be called a $C^{hv} -$ mixed trirecurrent Finsler space and denoted it by $C^{hv} - (M)TRF_n$ and the tensor is called $hv-$ mixed trirecurrent tensor and denoted it by $hv- (M)TR$.

Transvecting the condition (2.2) by g_{ih} and using (1.5b), (1.3c), (1.8a) and (1.8b), we get

$$(2.3) \quad C_{jhk|l|m|n} = c_{lmn} C_{jhk} + d_{lmn} (g_{kh} y_j - g_{jh} y_k) + f_{lm} (g_{kh} g_{jn} - g_{jh} g_{kn}).$$

Contracting the indies i and k in the condition (2.2), using (1.5f),(1.3a) and (1.3c,f), we get

$$(2.4) \quad C_{j|l|m|n} = c_{lmn} C_j + d_{lmn} (n-1) y_j + f_{lm} (n-1) g_{jn}.$$

Transvecting the condition (2.2) by g^{jk} and using (1.5d), (1.3d),(1.3b) and (1.8c,d), we get

$$(2.5) \quad C^i_{|l|m|n} = c_{lmn} C^i$$

Corollary 2.1: *In $C^{hv} - (M)TRF_n$ the tensors C_{jhk}, C_j, C^i are non– vanishing.*

The Cartan's first curvature tensor S^i_{jkh} satisfies the equation (1.10). Taking twice of h-covariant derivative and once of v −covariant derivative for (1.10) with respect to x^l, x^m and x^n respectively, we get,

$$(2.6) \; S^i_{jkh|l|m|n} = (C^i_{kr} C^r_{jh} - C^i_{rh} C^r_{jk})_{|l|m|n} \;,\;\; S^i_{jkh} \neq 0.$$

Or

$$S^i_{jkh|l|m|n} = C^i_{kr|l|m|n} C^r_{jh} + C^i_{kr|l|m} C^r_{jh|n} + C^i_{kr|l|n} C^r_{jh|m} + C^i_{kr|m|n} C^r_{jh|l}$$

$$+ C^i_{kr|n} C^r_{jh|l|m} + C^i_{kr|m} C^r_{jh|l|n} + C^i_{kr|l} C^r_{jh|m|n} + C^i_{kr} C^r_{jh|l|m|n} - C^i_{rh|l|m|n} C^r_{jk}$$

$$- C^i_{rh|l|m} C^r_{jk|n} - C^i_{rh|l|n} C^r_{jk|m} - C^i_{rh|m|n} C^r_{jk|l} - C^i_{rh|n} C^r_{jk|l|m} - C^i_{rh|m} C^r_{jk|l|n}$$

$$- C^i_{rh|l} C^r_{jk|m|n} - C^i_{rh} C^r_{jk|l|m|n}.$$

Using the condition (2.2) and (1.10) in the above equation, we get

$$(2.7) \; S^i_{jkh|l|m|n} = c_{lmn} S^i_{jkh} + 2 d_{lmn} C^i_{jh} y_k + b_{lm} (2 C^i_{jh} g_{kn} - \delta^i_k C_{jhn} - \delta^i_h C_{jkn})$$

$$[C^i_{kr|l|m} C^r_{jh|n} + C^i_{kr|l|n} C^r_{jh|m} + C^i_{kr|m|n} C^r_{jh|l} + C^i_{kr|n} C^r_{jh|l|m} + C^i_{kr|m} + C^i_{kr|l} C^r_{jh|m|n}$$

$$- C^i_{rh|l|n} C^r_{jk|m} - C^i_{rh|m|n} C^r_{jk|l} - C^i_{rh|n} C^r_{jk|l|m} - C^i_{rh|m} C^r_{jk|l|n} - C^i_{rh|l} C^r_{jk|m|n}.$$

This show that

$$(2.8) \; S^i_{jkh|l|m|n} = c_{lmn} S^i_{jkh} + 2 d_{lmn} C^i_{jh} y_k + b_{lm} (2 C^i_{jh} g_{kn} - \delta^i_k C_{jhn} - \delta^i_h C_{jkn})$$

if and only if

$$(2.9) \; [C^i_{kr|l|m} C^r_{jh|n} + C^i_{kr|l|n} C^r_{jh|m} + C^i_{kr|m|n} C^r_{jh|l} + C^i_{kr|n} C^r_{jh|l|m} + C^i_{kr|m} C^r_{jh|l|n}$$

$$+ C^i_{kr|l} C^r_{jh|m|n} - C^i_{rh|l|n} C^r_{jk|m} - C^i_{rh|m|n} C^r_{jk|l} - C^i_{rh|n} C^r_{jk|l|m} - C^i_{rh|m} C^r_{jk|l|n}$$

$$- C^i_{rh|l} C^r_{jk|m|n}] = 0$$

Transvecting (2.7) by g_{ip} , using(1.11a), (1.5b), (1.3c) and (1.8a,b),we get

$$(2.10) S_{jpkh|l|m|n} = c_{lmn} S_{jpkh} + 2 d_{lmn} C_{jph} + b_{lm} (2 C_{jph} g_{kn} - g_{kp} C_{jhn} - g_{hp} C_{jkn})$$

if and only if

(2.11) $[C^i_{kr\,|l|m}C^r_{jh\,|\,n} + C^i_{kr\,|l\,|\,n}C^r_{jh\,|m} + C^i_{kr\,|m\,|\,n}C^r_{jh\,|l} + C^i_{kr\,|\,n}C^r_{jh\,|l|m} + C^i_{kr\,|m}C^r_{jh\,|l\,|\,n}$

$$+ C^i_{kr\,|l}C^r_{jh\,|m\,|\,n} - C^i_{rh\,|l\,|\,n}C^r_{jk\,|m} - C^i_{rh\,|m\,|\,n}C^r_{jk\,|l} - C^i_{rh\,|\,n}C^r_{jk\,|l|m} - C^i_{rh\,|m}C^r_{jk\,|l\,|\,n}$$

$$-C^i_{rh\,|l}C^r_{jk\,|m\,|\,n}]g_{ip} = 0\ .$$

Contracting the indices i and h in (2.7), using (1.11b), (1.5f), (1.5b) and (1.3f), we get

(2.12) $S_{jk\,|l|m\,|\,n} = c_{lmn}S_{jk} + 2d_{lmn}C_j + b_{lm}\big(2C_jg_{kn} - (1+n)C_{jkn}\big) + [C^p_{kr\,|l|m}C^r_{jp\,|\,n}$

$$+C^p_{kr\,|l\,|\,n}C^r_{jp\,|m} + C^p_{kr\,|m\,|\,n}C^r_{jp\,|l} + C^p_{kr\,|\,n}C^r_{jp\,|l|m} + C^p_{kr\,|m}C^r_{jp\,|l\,|\,n} + C^p_{kr\,|l}C^r_{jp\,|m\,|\,n}$$

$$-C_{r\,|l\,|\,n}C^r_{jk\,|m} - C_{r\,|m\,|\,n}C^r_{jk\,|l} - C_{r\,|\,n}C^r_{jk\,|l|m} - C_{r\,|m}C^r_{jk\,|l\,|\,n} - C_{r\,|l}C^r_{jk\,|m\,|\,n}]$$

This show that

(2.13) $$S_{jk\,|l|m\,|\,n} = c_{lmn}S_{jk} + 2d_{lmn}C_j + b_{lm}\big(2C_jg_{kn} - (1+n)C_{jkn}\big)$$

if and only if

(2.14) $C^p_{kr\,|l|m}C^r_{jp\,|\,n} + C^p_{kr\,|l\,|\,n}C^r_{jp\,|m} + C^p_{kr\,|m\,|\,n}C^r_{jp\,|l} + C^p_{kr\,|\,n}C^r_{jp\,|l|m} + C^p_{kr\,|m}C^r_{jp\,|l\,|\,n}$

$$+C^p_{kr\,|l}C^r_{jp\,|m\,|\,n} - C_{r\,|l\,|\,n}C^r_{jk\,|m} - C_{r\,|m\,|\,n}C^r_{jk\,|l} - C_{r\,|\,n}C^r_{jk\,|l|m} - C_{r\,|m}C^r_{jk\,|l\,|\,n}$$

$$-C_{r\,|l}C^r_{jk\,|m\,|\,n} = 0$$

Theorem 2.1: *In $C^{hv} - (M)TRF_n$, the Cartan's first curvature tensor S^i_{jkh} , associate tensor S_{jpkh} and $S -$Ricci tensor S_{jk} of curvature tensor S^i_{jkh} are given by the equations (2.8), (2.10), and (2.13) if and only if the equations (2.9), (2.11) and (2.14) hold.*

Transvecting (2.12) by g^{ik}, using (1.11c), (1.2), (1.5d) and (1.8c,d), we get

(2.15 $$S^i_{j\,|l|m\,|\,n} = c_{lmn}S^i_j + 2d_{lmn}C_jg^{ik} + b_{lm}(2C_j\delta^i_n - (1+n)C^i_{jn})$$

if and only if

(2.16) $g^{ik}\{C^p_{kr\,|l|m}C^r_{jp\,|\,n} + C^p_{kr\,|l\,|\,n}C^r_{jp\,|m} + C^p_{kr\,|m\,|\,n}C^r_{jp\,|l} + C^p_{kr\,|\,n}C^r_{jp\,|l|m} + C^p_{kr\,|m}C^r_{jp\,|l\,|\,n}$

$$+C^p_{kr\,|l}C^r_{jp\,|m\,|\,n} - C_{r\,|l\,|\,n}C^r_{jk\,|m} - C_{r\,|m\,|\,n}C^r_{jk\,|l} - C_{r\,|\,n}C^r_{jk\,|l|m} - C_{r\,|m}C^r_{jk\,|l\,|\,n}$$

$$-C_{r\,|l}C^r_{jk\,|m\,|\,n}\} = 0.$$

Transvecting (2.12) by g^{jk}, using (1.11d), (1.5d,e), (1.2) and (1.8c,d), we get

(2.17) $$S_{|l|m\,|\,n} = c_{lmn}S + 2d_{lmn}C^k + (1-n)b_{lm}C_n$$

If and only if

(2.18) $g^{jk}\{C^p_{kr\,|l|m}C^r_{jp\,|\,n} + C^p_{kr\,|l\,|\,n}C^r_{jp\,|m} + C^p_{kr\,|m\,|\,n}C^r_{jp\,|l} + C^p_{kr\,|\,n}C^r_{jp\,|l|m} + C^p_{kr\,|m}C^r_{jp\,|l\,|\,n}$

$$+C^p_{kr\,|l}C^r_{jp\,|m\,|\,n} - C_{r\,|l\,|\,n}C^r_{jk\,|m} - C_{r\,|m\,|\,n}C^r_{jk\,|l} - C_{r\,|\,n}C^r_{jk\,|l|m} - C_{r\,|m}C^r_{jk\,|l\,|\,n}$$

$$-C_{r|l}C^{r}_{jk|m|n}\} = 0$$

Theorem 2.2: *In $C^{hv} − (M)TRF_n$, the deviation tensor S^i_j and curvature scalar S of curvature tensor S^i_{jkh} are non- vanishing if and only if (2.16) and (2.18) hold.*

<div align="center">

III. $P2 −$ Like -C^{hv} −Mixed Trirecurrent Space

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A$P2 −$ Like space is characterized by [8]

$$(3.1)\qquad P^i_{jkh} = \emptyset_j C^i_{kh} − \emptyset^i C_{jkh} ,$$

where $\emptyset_j$ and $\emptyset^i$ are nonzero covariant and contravariant verbs field, respectively.

Definition 3.1: *The $C^{hv} − (M)TRF_n$ which has the property of $P2 −$Like space [satisfies the condition (3.1)] called $P2 −$Like$−C^{hv} − (M)TRF_n$.*

Let us consider a $P2 −$Like$−C^{hv} − (M)TRF_n$.Now, taking $h −$covariant derivative for (3.1) second times with respect to x^l, x^m, successively, we get

$$(3.2)\qquad P^i_{jkh|l|m} = \emptyset_{j|l|m} C^i_{kh} + \emptyset_{j|l} C^i_{kh|m} + \emptyset_{j|m} C^i_{kh|l} + \emptyset_j C^i_{kh|l|m} −\emptyset^i_{|l|m} C_{jkh}$$

$$−\emptyset^i_{|l} C_{jkh|m} −\emptyset^i_{|m} C_{jkh|l} − \emptyset^i C_{jkh|l|m}.$$

Also, taking the $v −$ covariant derivative for (3.2) with respect to x^n, we get

$$(3.3)\qquad P^i_{jkh|l|m|n} = \emptyset_{j|l|m|n} C^i_{kh} + \emptyset_{j|l|m} C^i_{kh|n} + \emptyset_{j|l|n} C^i_{kh|m} + \emptyset_{j|m|n} C^i_{kh|l}$$

$$+\emptyset_{j|l} C^i_{kh|m|n} + \emptyset_{j|m} C^i_{kh|l|n} + \emptyset_{j|n} C^i_{kh|l|m} + \emptyset_j C^i_{kh|l|m|n} − \emptyset^i_{|l|m|n} C_{jkh}$$

$$−\emptyset^i_{|l|m} C_{jkh|n} − \emptyset^i_{|l|n} C_{jkh|m} − \emptyset^i_{|m|n} C_{jkh|l} − \emptyset^i_{|l} C_{jkh|m|n} − \emptyset^i_{|m} C_{jkh|l|n}$$

$$−\emptyset^i_{|n} C_{jkh|l|m} − \emptyset^i C_{jkh|l|m|n} .$$

Using the conditions (1.12),(1.13), (1.14),(1.15),(1.18),(1.19),(2.1),(2.2) and (2.3), we get

$$(3.4)\qquad P^i_{jkh|l|m|n} = \emptyset_{j|l|m|n} C^i_{kh} + \emptyset_{j|l|m}\left[\lambda_n C^i_{kh}\right]$$

$$+\emptyset_{j|l|n}\left[\lambda_m C^i_{kh} + \mu_m\left(\delta^i_h y_k − \delta^i_k y_h\right)\right] + \emptyset_{j|m|n}\left[\lambda_l C^i_{kh} + \mu_l\left(\delta^i_h y_k − \delta^i_k y_h\right)\right]$$

$$+\emptyset_{j|l}\left[a_{mn} C^i_{kh} + b_{mn}\left(\delta^i_h y_k − \delta^i_k y_h\right) + \mu_m\left(\delta^i_h g_{kn} − \delta^i_k g_{hn}\right)\right]$$

$$+\emptyset_{j|m}\left[a_{ln} C^i_{kh} + b_{ln}\left(\delta^i_h y_k − \delta^i_k y_h\right) + \mu_l\left(\delta^i_h g_{kn} − \delta^i_k g_{hn}\right)\right]$$

$$+\emptyset_{j|n}\left[a_{lm} C^i_{kh} + b_{lm}\left(\delta^i_h y_k − \delta^i_k y_h\right)\right] + \emptyset_j\left[c_{lmn} C^i_{kh} + d_{lmn}\left(\delta^i_h y_k − \delta^i_k y_h\right)\right.$$

$$+b_{lm}\left(\delta^i_h g_{kn} − \delta^i_k g_{hn}\right)] − \emptyset^i_{|l|m|n} C_{jkh} − \emptyset^i_{|l|m}\left[\lambda_n C_{jkh}\right] − \emptyset^i_{|l|n}\left[\lambda_m C_{jkh}\right.$$

$$+\mu_m\left(g_{hk} y_j − g_{jk} y_h\right)] − \emptyset^i_{|m|n}\left[\lambda_l C_{jkh} + \mu_l\left(g_{hk} y_j − g_{jk} y_h\right)\right]$$

$$−\emptyset^i_{|l}\left[a_{mn} C_{jkh} + b_{mn}\left(g_{hk} y_j − g_{jk} y_h\right) + \mu_l\left(g_{hk} g_{jn} − g_{jk} g_{hn}\right)\right]$$

$$− \emptyset^i_{|m}\left[a_{ln} C_{jkh} + b_{ln}\left(g_{hk} y_j − g_{jk} y_h\right) + \mu_l\left(g_{hk} g_{jn} − g_{jk} g_{hn}\right)\right]$$

$$-\phi^i_{\,|\,n}\big[a_{lm}C_{jkh} + b_{lm}\big(g_{hk}y_j - g_{jk}y_h\big)\big]$$

$$-\phi^i\big[c_{lmn}C_{jkh} + {}+d_{lmn}\big(g_{hk}y_j - g_{jk}y_h\big) + b_{lm}\big(g_{hk}g_{jn} - g_{jk}g_{hn}\big)\big]\,.$$

Using the condition (3.1) in the condition (3.4), we get

$$(3.5)\quad P^i_{jkh|l\,|\,n} = c_{lmn}P^i_{jkh} + d_{lmn}\phi_j\big(\delta^i_h y_k - \delta^i_k y_h\big)$$

if and only if

$$(3.6)\quad \phi_{j|l|m\,|\,n}\,C^i_{kh} + \phi_{j\,|l|m}\big[\lambda_n C^i_{kh}\big] + \phi_{j|l\,|\,n}\big[\lambda_m\,C^i_{kh} + \mu_m\big(\delta^i_h y_k - \delta^i_k y_h\big)\big]$$

$$+\phi_{j\,|m\,|\,n}\big[\lambda_l\,C^i_{kh} + \mu_l\big(\delta^i_h y_k - \delta^i_k y_h\big)\big]$$

$$+\phi_{j|l}\big[a_{mn}C^i_{kh} + b_{mn}\big(\delta^i_h y_k - \delta^i_k y_h\big) + \mu_m\big(\delta^i_h g_{kn} - \delta^i_k g_{hn}\big)\big]$$

$$+\phi_{j|m}\big[a_{ln}C^i_{kh} + b_{ln}\big(\delta^i_h y_k - \delta^i_k y_h\big) + \mu_l\big(\delta^i_h g_{kn} - \delta^i_k g_{hn}\big)\big]$$

$$+\phi_{j\,|\,n}\big[a_{lm}C^i_{kh} + b_{lm}\big(\delta^i_h y_k - \delta^i_k y_h\big)\big] + \phi_j\big[\,c_{lmn}C^i_{kh} + d_{lmn}\big(\delta^i_h y_k - \delta^i_k y_h\big)$$

$$+b_{lm}\big(\delta^i_h g_{kn} - \delta^i_k g_{hn}\big)\big] - \phi^i_{|l|m\,|\,n}\,C_{jkh} - \phi^i_{|l|m}\big[\lambda_n C_{jkh}\big] - \phi^i_{|l\,|\,n}\big[\lambda_m C_{jkh}$$

$$+\mu_m\big(\,g_{hk}y_j - g_{jk}y_h\big)\big] - \phi^i_{|m\,|\,n}\big[\lambda_l C_{jkh} + \mu_l\big(\,g_{hk}y_j - g_{jk}y_h\big)\big]$$

$$-\phi^i_{|l}\big[a_{mn}C_{jkh} + b_{mn}\big(g_{hk}y_j - g_{jk}y_h\big) + \mu_m\big(g_{hk}g_{jn} - g_{jk}g_{hn}\big)\big]$$

$$-\phi^i_{|m}\big[a_{ln}C_{jkh} + b_{ln}\big(g_{hk}y_j - g_{jk}y_h\big) + \mu_l\big(g_{hk}g_{jn} - g_{jk}g_{hn}\big)\big]$$

$$-\phi^i_{\,|\,n}\big[a_{lm}C_{jkh} + b_{lm}\big(g_{hk}y_j - g_{jk}y_h\big)\big] - \phi^i\big[c_{lmn}C_{jkh}$$

$$+d_{lmn}\big(g_{hk}y_j - g_{jk}y_h\big) + b_{lm}\big(g_{hk}g_{jn} - g_{jk}g_{hn}\big)\big] = 0\,.$$

Transvecting (3.4) by g_{ir}, using (1.9b),(1.3c), (1.5b),(1.8a) and (1.8b), we get

$$(3.7)\quad P_{jrkh|l|m\,|\,n} = c_{lmn}P_{jrkh} + d_{lmn}\phi_j\big(g_{hr}y_k - g_{kr}y_h\big)$$

if and only if

$$(3.8)\quad g_{ir}\big\{\phi_{j|l|m}\big[\lambda_n C^i_{kh}\big] + \phi_{j|l\,|\,n}\big[\lambda_m\,C^i_{kh} + \mu_m\big(\delta^i_h y_k - \delta^i_k y_h\big)\big]$$

$$+\phi_{j|m\,|\,n}\big[\lambda_l\,C^i_{kh} + \mu_l\big(\delta^i_h y_k - \delta^i_k y_h\big)\big] + \phi_{j|l}\big[a_{mn}C^i_{kh} + b_{mn}\big(\delta^i_h y_k - \delta^i_k y_h\big)$$

$$+\mu_m\big(\delta^i_h g_{kn} - \delta^i_k g_{hn}\big)\big] + \phi_{j|m}\big[a_{ln}C^i_{kh} + b_{ln}\big(\delta^i_h y_k - \delta^i_k y_h\big)$$

$$+\mu_l\big(\delta^i_h g_{kn} - \delta^i_k g_{hn}\big)\big] + \phi_{j\,|\,n}\big[a_{lm}C^i_{kh} + b_{lm}\big(\delta^i_h y_k - \delta^i_k y_h\big)\big]$$

$$+\phi_j\big[\,b_{lm}\big(\delta^i_h g_{kn} - \delta^i_k g_{hn}\big)\big] - \phi^i_{|l|m\,|\,n}\,C_{jkh} - \phi^i_{|l|m}\big[\lambda_n C_{jkh}\big]$$

$$-\phi^i_{|l\,|\,n}\big[\lambda_m C_{jkh} + \mu_m\big(\,g_{hk}y_j - g_{jk}y_h\big)\big] - \phi^i_{|m\,|\,n}\big[\lambda_l C_{jkh} + \mu_l\big(\,g_{hk}y_j - g_{jk}y_h\big)\big]$$

$$-\phi^i_{|l}\big[a_{mn}C_{jkh} + b_{mn}\big(g_{hk}y_j - g_{jk}y_h\big) + \mu_m\big(g_{hk}g_{jn} - g_{jk}g_{hn}\big)\big]$$

$$- \varnothing^i_{|m}[a_{ln}C_{jkh} + b_{ln}(g_{hk}y_j - g_{jk}y_h) + \mu_l(g_{hk}g_{jn} - g_{jk}g_{hn})]$$

$$-\varnothing^i_{|n}[a_{lm}C_{jkh} + b_{lm}(g_{hk}y_j - g_{jk}y_h)] - \varnothing^i[d_{lmn}(g_{hk}y_j - g_{jk}y_h)$$

$$+b_{lm}(g_{hk}g_{jn} - g_{jk}g_{hn})]\} = 0 .$$

Theorem 3.1: *In* $P2 -Like-C^{hv} - (M)TRF_n$ *, the mixed* $hv -$ *covariant derivative of third order for Cartan's second curvature tensor* P^i_{jkh} *and associate* P_{jrkh} *given by (3.5) and (3.7),respectively, if and only if (3.6) and (3.8) hold.*

Transvecting the condition (3.4) by y^j, using (1.1a),(1.1b),(1.9a), (1.8e) and (1.8f), we get

$$(3.9) \qquad P^i_{kh|l|m|n} = c_{lmn} P^i_{kh} + + \varnothing \, d_{lmn}(\delta^i_h y_k - \delta^i_k y_h)$$

if and only if

$$(3.10) \quad P^i_{jkh|l|m|n} y^j + P^i_{jkh|l|m} y^j_{|n} +y^j \{\varnothing_{j|l|m|n} \, C^i_{kh} +\varnothing_{j|l|m}[\lambda_n C^i_{kh}]$$

$$+\varnothing_{j|l|n}[\lambda_m \, C^i_{kh} + \mu_m(\delta^i_h y_k - \delta^i_k y_h)] + \varnothing_{j|m|n}[\lambda_l \, C^i_{kh} + \mu_l(\delta^i_h y_k - \delta^i_k y_h)]$$

$$+\varnothing_{j|l}[a_{mn} C^i_{kh} + b_{mn}(\delta^i_h y_k - \delta^i_k y_h) + \mu_m(\delta^i_h g_{kn} - \delta^i_k g_{hn})]$$

$$+\varnothing_{j|m}[a_{ln} C^i_{kh} + b_{ln}(\delta^i_h y_k - \delta^i_k y_h) + \mu_l(\delta^i_h g_{kn} - \delta^i_k g_{hn})]$$

$$+\varnothing_{j|n}[a_{lm} C^i_{kh} + b_{lm}(\delta^i_h y_k - \delta^i_k y_h)] + \varnothing_j[b_{lm}(\delta^i_h g_{kn} - \delta^i_k g_{hn})]$$

$$-\varnothing^i_{|l|m|n} \, C_{jkh} - \varnothing^i_{|l|m}[\lambda_n C_{jkh}] - \varnothing^i_{|l|n}[\lambda_m C_{jkh} + \mu_m(g_{hk}y_j - g_{jk}y_h)]$$

$$-\varnothing^i_{|m|n}[\lambda_l C_{jkh} + \mu_l(g_{hk}y_j - g_{jk}y_h)]$$

$$-\varnothing^i_{|l}[a_{mn} C_{jkh} + b_{mn}(g_{hk}y_j - g_{jk}y_h) + \mu_m(g_{hk}g_{jn} - g_{jk}g_{hn})]$$

$$- \varnothing^i_{|m}[a_{ln} C_{jkh} + b_{ln}(g_{hk}y_j - g_{jk}y_h) + \mu_l(g_{hk}g_{jn} - g_{jk}g_{hn})]$$

$$-\varnothing^i_{|n}[a_{lm} C_{jkh} + b_{lm}(g_{hk}y_j - g_{jk}y_h)] - \varnothing^i[d_{lmn}(g_{hk}y_j - g_{jk}y_h)$$

$$+b_{lm}(g_{hk}g_{jn} - g_{jk}g_{hn})]\} = 0,$$

where $y^j \varnothing_j = \varnothing$.Thus, we conclude

Theorem 3.2: *In* $P2 -Like-C^{hv} - (M)TRF_n$ *, the mixed* $hv -$*covariant derivative of third order for* $v(hv) -$ *torsion tensor* P^i_{kh} *given by (3.9) if and only if (3.10) holds.*

Contracting the indices i and $h$in the condition (3.4), using (1.9c),(1.5f),(1.3a,f) and (1.3c), we get

$$(3.11) \quad P_{jk|l|m|n} = c_{lmn} P_{jk} + \varnothing_j \, d_{lmn}(n-1)y_k$$

if and only if

$$(3.12) \quad \varnothing_{j|l|m|n} C_k + + \varnothing_{j|l|m}[\lambda_n C_k]+\varnothing_{j|l|n}[\lambda_m C_k + \mu_m(n-1)y_k]$$

$$+\varnothing_{j|m|n}[\lambda_l C_k + \mu_l(n-1)y_k]+\varnothing_{j|l}[a_{mn} C_k + b_{mn}(n-1)y_k$$

$$+\mu_m(n-1)g_{kn}] + \varnothing_{j|m}[a_{ln} C_k + b_{ln}(n-1)y_k + \mu_l(n-1)g_{kn}]$$

$$+\emptyset_{j\,|\,n}[a_{lm}\,C_k + b_{lm}\,(n-1)y_k] + \emptyset_j\,[\,b_{lm}\,(n-1)g_{kn}]$$

$$-\emptyset^r_{|l|m\,|\,n}\,C_{jkr} - \emptyset^r_{|l|m}[\lambda_n\,C_{jkr}] - \emptyset^r_{|l\,|\,n}[\lambda_m\,C_{jkr} + \mu_m\,(\,g_{hk}\,y_j\, -\, g_{jk}\,y_h)]$$

$$-\emptyset^r_{|m\,|\,n}[\lambda_l\,C_{jkr}\, +\, \mu_l\,(\,g_{hk}\,y_j\, -\, g_{jk}\,y_h)]$$

$$-\emptyset^r_{|l}[a_{mn}\,C_{jkr}\, +\, b_{mn}\,(\,g_{hk}\,y_j\, -\, g_{jk}\,y_h) + \mu_m\,(g_{hk}\,g_{jn}\, -\, g_{jk}\,g_{hn})]$$

$$-\,\emptyset^r_{|m}[a_{ln}\,C_{jkr}\, +\, b_{ln}\,(\,g_{hk}\,y_j\, -\, g_{jk}\,y_h) + \mu_l\,(g_{hk}\,g_{jn}\, -\, g_{jk}\,g_{hn})]$$

$$-\emptyset^r_{|\,n}[a_{lm}\,C_{jkr}\, +\, b_{lm}\,(\,g_{hk}\,y_j\, -\, g_{jk}\,y_h)] - \emptyset^r[d_{lmn}\,(\,g_{rk}\,y_j\, -\, g_{jk}\,y_r)$$

$$+b_{lm}\,(g_{rk}\,g_{jn}\, -\, g_{jk}\,g_{rn})] = 0\ .$$

Contracting the indices i and h in the condition (3.9) and (3.10) using (1.9c,d),(1.5f), (1.3a,f) and (1.3c), we get

$$(3.13)\quad P_{k|l|m\,|\,n} = c_{lmn}\,P_k + \emptyset\,d_{lmn}\,(\,n-1)y_k$$

if and only if

$$(3.14)\quad P_{jk\,|l|m\,|\,n}\,y^j + P_{jk\,|l|m}\,y^j_{|\,n} + y^j\,\{\emptyset_{j|l|m\,|\,n}\,C_k + \emptyset_{j\,|l|m}[\lambda_n\,C_k]$$

$$+\emptyset_{j\,|\,l\,|\,n}[\lambda_m\,C_k + \mu_m\,(\,n-1)y_k] + \emptyset_{j\,|m\,|\,n}[\lambda_l\,C_k + \mu_l\,(\,n-1)y_k]$$

$$+\emptyset_{j\,|l}[a_{mn}\,C_k\, +\, b_{mn}\,(\,n-1)y_k + \mu_m\,(n-1)g_{kn}]$$

$$+\emptyset_{j\,|m}[a_{ln}\,C_k\, +\, b_{ln}\,(\,n-1)y_k + \mu_l\,(n-1)g_{kn}]$$

$$+\emptyset_{j\,|\,n}[a_{lm}\,C_k + b_{lm}\,(\,n-1)y_k] + \emptyset_j\,[\,b_{lm}\,(n-1)g_{kn}]$$

$$-\emptyset^r_{|l|m\,|\,n}\,C_{jkr} - \emptyset^r_{|l|m}[\lambda_n\,C_{jkr}] - \emptyset^r_{|l\,|\,n}[\lambda_m\,C_{jkr} + \mu_m\,(\,g_{rk}\,y_j\, -\, g_{jk}\,y_r)]$$

$$-\emptyset^r_{|m\,|\,n}[\lambda_l\,C_{jkr}\, +\, \mu_l\,(\,g_{rk}\,y_j\, -\, g_{jk}\,y_r)]$$

$$-\emptyset^r_{|l}[a_{mn}\,C_{jkr}\, +\, b_{mn}\,(\,g_{rk}\,y_j\, -\, g_{jk}\,y_r) + \mu_m\,(g_{rk}\,g_{jn}\, -\, g_{jk}\,g_{rn})]$$

$$-\,\emptyset^r_{|m}[a_{ln}\,C_{jkr}\, +\, b_{ln}\,(\,g_{rk}\,y_j\, -\, g_{jk}\,y_r) + \mu_l\,(g_{rk}\,g_{jn}\, -\, g_{jk}\,g_{rn})]$$

$$-\emptyset^r_{|\,n}[a_{lm}\,C_{jkr}\, +\, b_{lm}\,(\,g_{rk}\,y_j\, -\, g_{jk}\,y_r)] - \emptyset^r[d_{lmn}\,(\,g_{rk}\,y_j\, -\, g_{jk}\,y_r)$$

$$+b_{lm}\,(g_{rk}\,g_{jn}\, -\, g_{jk}\,g_{rn})]\} = 0\ .$$

Thus, we conclude

Theorem 3.3: *In $P2-Like-C^{hv}-(M)TRF_n$, the mixed hv $-$covariant derivative of third order for the $P-Ricci$ tensor P_{jk} and the curvature vector P_k are non- vanishing if and only if (3.11) and (3.13) hold.*

IV. $P^*-C^{hv}-$Mixed Trirecurrent Space

A P^*- Finsler space is characterized by [8]

$$(4.1)\quad P^i_{kh} = C^i_{kh|j}\,y^j = \emptyset C^i_{kh}\ ,\quad \emptyset \neq 0,$$

Where $\quad P^i_{jkh}\,y^j = P^i_{kh} = C^i_{kh|j}\,y^j$.

Definition 4.1: *The $GC^{hv} - (M)TRF_n$ which has the property of $P2 -$Like space [satisfies the condition (4.1)] called $P^* - C^{hv} - (M)TRF_n$.*

Remark 4.1: All results which obtained in $P2 -$Like$- C^{hv} - (M)TRF_n$ satisfy in $P^* - C^{hv} - (M)TRF_n$.

Let us consider an $P^* - C^{hv} - (M)TRF_n$. Now, taking the $h-$covariant derivative for (4.1) second times with respect to x^l, x^m, successively, we get

$$(4.2) \quad P^i_{kh|l|m} = \emptyset_{|l|m} C^i_{kh} + \emptyset_{|l} C^i_{kh|m} + \emptyset_{|m} C^i_{kh|l} + \emptyset\, C^i_{kh|l|m}$$

Also, taking the $v-$covariant derivative for (4.2) with respect to x^n, we get

$$(4.3) \quad P^i_{kh|l|m|n} = \emptyset_{|l|m|n} C^i_{kh} + \emptyset_{|l|m} C^i_{kh|n} + \emptyset_{|l|n} C^i_{kh|m} + \emptyset_{|l} C^i_{kh|m|n}$$

$$+\emptyset_{|m|n} C^i_{kh|l} + \emptyset_{|m} C^i_{kh|l|n} + \emptyset_{|n} C^i_{kh|l|m} + \emptyset\, C^i_{kh|l|m|n}\,.$$

Using (1.12),(1.14),(1.18),(2.1) and the condition (2.2) in (4.3), we get

$$(4.4) \quad P^i_{kh|l|m|n} = \emptyset_{|l|m|n} C^i_{kh} + \emptyset_{|l|m}[\lambda_n C^i_{kh}] + \emptyset_{|l|n}[\lambda_m C^i_{kh} + \mu_m(\delta^i_h y_k - \delta^i_k y_h)]$$

$$+\emptyset_{|l}[a_{mn} C^i_{kh} + b_{mn}(\delta^i_h y_k - \delta^i_k y_h) + \mu_m(\delta^i_h g_{kn} - \delta^i_k g_{hn})]$$

$$+\emptyset_{|m|n}[\lambda_l C^i_{kh} + \mu_l(\delta^i_h y_k - \delta^i_k y_h)] + \emptyset_{|m}[a_{ln} C^i_{kh} + b_{ln}(\delta^i_h y_k - \delta^i_k y_h)$$

$$+\mu_l(\delta^i_h g_{kn} - \delta^i_k g_{hn})] + \emptyset_{|n}[a_{lm} C^i_{kh} + b_{lm}(\delta^i_h y_k - \delta^i_k y_h)]$$

$$+\emptyset[c_{lmn} C^i_{kh} + d_{lmn}(\delta^i_h y_k - \delta^i_k y_h) + b_{lm}(\delta^i_h g_{kn} - \delta^i_k g_{hn})].$$

Using the condition (4.1) in the (4.4), we get

$$(4.5) \quad P^i_{kh|l|m|n} = c_{lmn} P^i_{kh} + f_{lmn}(\delta^i_h y_k - \delta^i_k y_h) + \emptyset_{|l|m|n} C^i_{kh} + \emptyset_{|l|m}[\lambda_n C^i_{kh}]$$

$$+\emptyset_{|l|n}[\lambda_m C^i_{kh} + \mu_m(\delta^i_h y_k - \delta^i_k y_h)]$$

$$+\emptyset_{|l}[a_{mn} C^i_{kh} + b_{mn}(\delta^i_h y_k - \delta^i_k y_h) + \mu_m(\delta^i_h g_{kn} - \delta^i_k g_{hn})]$$

$$+\emptyset_{|m|n}[\lambda_l C^i_{kh} + \mu_l(\delta^i_h y_k - \delta^i_k y_h)] + \emptyset_{|m}[a_{ln} C^i_{kh} + b_{ln}(\delta^i_h y_k - \delta^i_k y_h)$$

$$+\mu_l(\delta^i_h g_{kn} - \delta^i_k g_{hn})] + \emptyset_{|n}[a_{lm} C^i_{kh} + b_{lm}(\delta^i_h y_k - \delta^i_k y_h)]$$

$$+\emptyset[b_{lm}(\delta^i_h g_{kn} - \delta^i_k g_{hn})],$$

where $f_{lmn} = \emptyset d_{lmn}$. This show that

$$(4.6) \quad P^i_{kh|l|m|n} = c_{lmn} P^i_{kh} + f_{lmn}(\delta^i_h y_k - \delta^i_k y_h)$$

if and only if

$$(4.7) \quad \emptyset_{|l|m|n} C^i_{kh} + \emptyset_{|l|m}[\lambda_n C^i_{kh}] + \emptyset_{|l|n}[\lambda_m C^i_{kh} + \mu_m(\delta^i_h y_k - \delta^i_k y_h)]$$

$$+\emptyset_{|l}[a_{mn} C^i_{kh} + b_{mn}(\delta^i_h y_k - \delta^i_k y_h) + \mu_m(\delta^i_h g_{kn} - \delta^i_k g_{hn})]$$

$$+\emptyset_{|m|n}[\lambda_l C^i_{kh} + \mu_l(\delta^i_h y_k - \delta^i_k y_h)] + \emptyset_{|m}[a_{ln} C^i_{kh} + b_{ln}(\delta^i_h y_k - \delta^i_k y_h)]$$

$$+\mu_l(\delta_h^i g_{kn} - \delta_k^i g_{hn})] + \emptyset_{|n}[a_{lm}C_{kh}^i + b_{lm}(\delta_h^i y_k - \delta_k^i y_h)]$$

$$+\emptyset[b_{lm}(\delta_h^i g_{kn} - \delta_k^i g_{hn})] = 0.$$

Transvecting the condition (4.5) by g_{ir}, using (1.9e),(1.5b),(1.3c) and (1.8a,b), we get

$$(4.8) \quad P_{rkh|l|m|n} = c_{lmn}P_{rkh} + f_{lmn}(g_{hr}y_k - g_{hr}y_h)$$

if and only if

$$(4.9) \quad \emptyset_{|l|m|n}C_{krh} + \emptyset_{|l|m}[\lambda_n C_{krh}] + \emptyset_{|l|n}[\lambda_m C_{krh} + \mu_m(g_{hr}y_k - g_{kr}y_h)]$$

$$+\emptyset_{|l}[a_{mn}C_{krh} + b_{mn}(g_{hr}y_k - g_{kr}y_h) + \mu_m(g_{hr}g_{kn} - g_{kr}g_{hn})]$$

$$+\emptyset_{|m|n}[\lambda_l C_{krh} + \mu_l(g_{hr}y_k - g_{kr}y_h)] + \emptyset_{|m}[a_{ln}C_{krh} + b_{ln}(g_{hr}y_k - g_{kr}y_h)$$

$$+\mu_l(g_{hr}g_{kn} - g_{kr}g_{hn})] + \emptyset_{|n}[a_{lm}C_{krh} + b_{lm}(g_{hr}y_k - g_{kr}y_h)]$$

$$+\emptyset[b_{lm}(g_{hr}g_{kn} - g_{kr}g_{hn})] = 0.$$

Theorem 4.1: *In $P^* - C^{hv} - (M)TRF_n$, the mixed $hv -$ covariant derivative of third order for the $v(hv) -$ torsion tensor P_{kh}^i and its associate P_{rkh} given by (4.6)and (4.8), respectively if and only if (4.7) and (4.9),respectively.*

Contracting the indices i and h in (4.5), using (1.9d),(1.3f,a),(1.5f)and (1.3c), we get

$$(4.10) \quad P_{k|l|m|n} = c_{lmn}P_k + f_{lmn}(n-1)y_k$$

if and only if

$$(4.11) \quad \emptyset_{|l|m|n}C_k + \emptyset_{|l|m}[\lambda_n C_k] + \emptyset_{|l|n}[\lambda_m C_k + \mu_m(n-1)y_k]$$

$$+\emptyset_{|l}[a_{mn}C_k + b_{mn}(n-1)y_k + \mu_m(n-1)g_{kn}]$$

$$+\emptyset_{|m|n}[\lambda_l C_k + \mu_l(n-1)y_k] + \emptyset_{|m}[a_{ln}C_k + b_{ln}(n-1)y_k + \mu_l(n-1)g_{kn}]$$

$$+\emptyset_{|n}[a_{lm}C_k + b_{lm}(n-1)y_k] + \emptyset[b_{lm}(n-1)g_{kn}] = 0.$$

Thus, we conclude

Theorem 4.2: *In $P^* - C^{hv} - (M)TRF_n$, the curvature vector P_k is non- vanishing if and only if (4.11) hold.*

V. Conclusion

We studied certain types of a $C^{hv} -$mixed trirecurrent, such as $P2 -$ Like space and $P^* -$space, and derived new results characterizing these spaces in the main space**.**

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