International Research Journal of Innovations in Engineering and Technology (IRJIET)

ISSN (online): 2581-3048

Volume 10, Issue 6, pp 55-59, June-2026

https://doi.org/10.47001/IRJIET/2026.106005

Forward Dynamic Analysis of a Spatial Four-Bar Linkage

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***Abstract* - This paper presents an approach for the inverse dynamic analysis of a spatial four-bar linkage mechanism. By employing the Lagrange multiplier formulation with redundant generalized coordinates for closed-loop mechanical systems, expressed in both analytical and matrix forms, and combining it with the orthogonal projection method for establishing coordinate systems, the equations of motion of the mechanism are derived. Based on these equations, the driving torque applied to the input link is determined. The obtained results can be used for the analysis and design of mechanical systems with similar configurations, such as shaper mechanisms, hydraulic presses, and related machinery.**

Keywords: inverse dynamics; spatial linkage mechanism; Lagrange multiplier formulation; orthogonal projection method; air compressor.

I. INTRODUCTION

Dynamic analysis is one of the primary problems that must be solved in the study of mechanical systems, providing a foundation for addressing practical engineering issues such as **strength evaluation**, **machine balancing**, and **optimization**.

Numerous methods for dynamic analysis have been developed and applied worldwide. Nevertheless, most existing methods are mathematically complex, especially when dealing with **spatial mechanisms**, where the complexity of modeling and computation increases considerably. In this paper, an **orthogonal projection method** is proposed for the kinematic and dynamic analysis of spatial mechanisms. The proposed approach aims to simplify the formulation and solution of dynamic analysis problems, thereby improving computational efficiency and facilitating practical engineering applications.

1.1 Theoretical Basis for Dynamic Analysis of Spatial Mechanisms.

Theorem 1 (Van Khang, 2010). Let $A(x) \in \mathbb{R}^{m\times p}$ be a matrix-valued function of the vector $x \in \mathbb{R}^n$, and let $x = x(t)$ be a function of time. Then, the time derivative of matrix A is determined by:

$$\frac{d}{dt}A(x) = \frac{\partial A(x)}{\partial x}\dot{x} \qquad \textbf{(1)}$$

Theorem 2 (Van Khang, 2010). Let $A(x) \in \mathbb{R}^{m\times p}$ and $B(x) \in \mathbb{R}^{q\times s}$ be matrix-valued functions of the vector $x \in \mathbb{R}^n$. Then, the partial derivative of the product of the two matrices is given by:

$$\frac{\partial}{\partial x}(A(x)B(x)) = \frac{\partial A(x)}{\partial x} \otimes B(x) + A(x) \otimes \frac{\partial B(x)}{\partial x} \qquad \textbf{(2)}$$

where I_n is the identity matrix of order n.

The **Lagrange multiplier formulation** is used to establish the system of differential equations of motion for mechanical systems with **holonomic constraints** (constraint equations independent of velocities) and **closed-loop structures** using redundant generalized coordinates:

$$M(q)\ddot{q} + C(q,\dot{q}) + F(q) + \Phi_q^T\lambda = \tau \qquad \textbf{(3)}$$

where:

- f— number of degrees of freedom of the mechanical system;
- n— number of generalized coordinates ($n > f$);
- $q = [q_1, q_2, \dots, q_n]^T$— generalized coordinate vector;
- $\dot{q} = [\dot{q}_1, \dot{q}_2, \dots, \dot{q}_n]^T$— generalized velocity vector;
- $\ddot{q} = [\ddot{q}_1, \ddot{q}_2, \dots, \ddot{q}_n]^T$— generalized acceleration vector;
- $\alpha = n - f$— number of additional constraint equations;
- $f_i(q) = 0,\ i = 1,2, \dots, \alpha$— additional constraint equations;

$$\Phi_q = \frac{\partial f}{\partial q}$$

is the Jacobian matrix of the constraint equations with respect to the generalized coordinates,

where,

$$f = [f_1, f_2, \dots, f_\alpha]^T$$

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$$\lambda = [\lambda_1, \lambda_2, \dots, \lambda_\alpha]^T$$

is the vector of **Lagrange multipliers**.

1.2 Kinematic Analysis of Spatial Mechanisms Using the Orthogonal Projection Method.

The objective of this method is to apply the **orthogonal projection technique** together with **redundant generalized coordinates** to formulate the constraint equations of spatial mechanisms. **Figure 3.1** illustrates the procedure of the orthogonal projection method for dynamic analysis of spatial mechanisms.

Notes:

- When selecting the fixed coordinate system $Oxyz$, the following principles should be considered: If the link connected to the frame rotates about a fixed axis, the xOyplane should be chosen parallel to the axis of rotation. If the link connected to the frame undergoes translational motion, one of the two axes, Oxor Oy, should be selected parallel to the direction of translation.
- For each link, it is necessary to determine the angles between the projections of the two coordinate axes attached to the link and the Ox(or Oy) axis, as well as the Ozaxis. In the case of a prismatic joint, an additional generalized coordinate corresponding to the translational displacement must be introduced.

Procedure of the Orthogonal Projection Method for Kinematic and Dynamic Analysis of Spatial Mechanisms

- Establish the mechanism model
- Select the fixed coordinate system $Oxyz$
  Select coordinate systems attached to the links
- Project the mechanism and coordinate axes onto the xOyplane
- Choose the generalized coordinates
- Formulate the vector constraint equations
  Project the constraint equations onto the fixed coordinate axes
- Determine the direction cosine matrix of each link
  Establish expressions for the position, velocity, acceleration, etc. of each link

II. DYNAMIC ANALYSIS OF A SPATIAL CRANK–SLIDER MECHANISM

2.1 Model of a Spatial Crank–Slider Mechanism

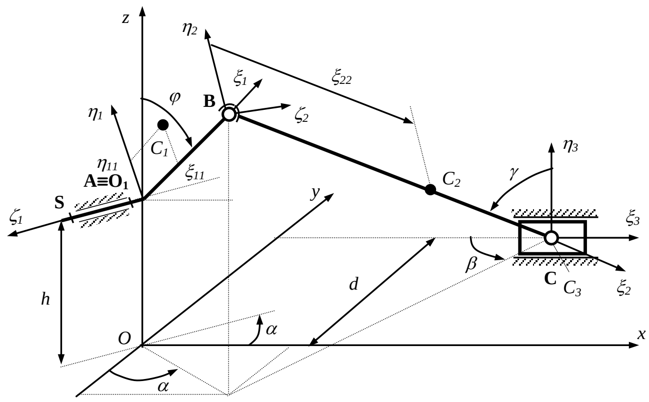

Figure 2.1: Model of a Spatial Crank–Slider Mechanism

Consider a **spatial crank–slider mechanism** with mass, mass moment of inertia, link length, and center-of-mass position defined as follows:

- **Link 1** rotates about a fixed axis and is characterized by:
- **Link 2** undergoes planar motion and is modeled as two concentrated masses located at its two ends:
- **Link 3** performs translational motion and is characterized by:

The redundant generalized coordinates are chosen as:

$$s = \begin{bmatrix} \varphi & \beta & \gamma & x_C \end{bmatrix}^T \qquad (2.1)$$

The constraint equations are given by:

$$\begin{aligned} f_1 &= l_1 \sin\varphi \sin\alpha + l_2 \sin\gamma \cos\beta - x_C = 0 \\ f_2 &= -l_1 \sin\varphi \cos\alpha + l_2 \sin\gamma \sin\beta - d = 0 \qquad (2.2) \\ f_3 &= h + l_1 \cos\varphi - l_2 \cos\gamma = 0 \end{aligned}$$

2.2 Derivation of the Differential Equations of Motion of the Mechanism

To simplify the analysis, the mass of the connecting rod **BD** is replaced by two concentrated masses located at points **B** and **C**. Accordingly, the kinetic energy of the mechanism can be expressed as

$$T = T_1 + T_2 + T_3 \quad (2.3)$$

where $T_1$is the kinetic energy of the driving link **SAB**. Since link **SAB** rotates about the fixed axis **SA**, its kinetic energy is given by

ISSN (online): 2581-3048
Volume 10, Issue 6, pp 55-59, June-2026
https://doi.org/10.47001/IRJIET/2026.106005

$$T_1 = \frac{1}{2} J_1 \dot{\varphi}^2 \qquad (2.4)$$

is the kinetic energy of link **BC**. Link **BC** is modeled as two concentrated masses located at its endpoints. Therefore, half of the mass attached to point **B** is considered to undergo rotational motion about the fixed axis **SA**, while the other half attached to point **C** is assumed to undergo translational motion:

$$T_2 = T_2^{(1)} + T_2^{(2)} = \frac{1}{2}\left(\frac{m_2}{2} l_1^2 \dot{\varphi}^2\right) + \frac{1}{2}\left(\frac{m_2}{2} \dot{x}_C^2\right) \qquad (2.5)$$

The slider **D** performs translational motion; therefore, its kinetic energy is written as

$$T_3 = \frac{1}{2}\left(m_3 \dot{x}_C^2\right) \qquad (2.6)$$

Substituting Eqs. (2.89), (2.90), and (2.91) into Eq. (2.88), the total kinetic energy of the mechanism is obtained as

$$\begin{aligned} T &= T_1 + T_2 + T_3 \\ &= \frac{1}{2} J_1 \dot{\varphi}^2 + \frac{1}{2}\left(\frac{m_2}{2} l_1^2 \dot{\varphi}^2\right) + \frac{1}{2}\left(\frac{m_2}{2} \dot{x}_C^2\right) + \frac{1}{2}\left(m_3 \dot{x}_C^2\right) \\ &= \frac{1}{2}\left(J_1 + \frac{m_2}{2} l_1^2\right)\dot{\varphi}^2 + \frac{1}{2}\left(\frac{m_2}{2} + m_3\right)\dot{x}_C^2 \end{aligned} \qquad (2.7)$$

Hence, the generalized mass matrix is determined as

$$\mathbf{M}(\mathbf{s}) = \begin{bmatrix} J_1 + \frac{m_2}{2} l_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{m_2}{2} + m_3 \end{bmatrix} \qquad (2.8)$$

From this, the centrifugal and Coriolis inertia matrix can be obtained.

The potential energy of the system is expressed as

$$\Pi = m_1 g\left(a_1 \cos\varphi + h\right) + \frac{m_2}{2} g\left(l_1 \cos\varphi + h\right) \Rightarrow \mathbf{g}(\mathbf{s}) = \begin{bmatrix} \left(m_1 a_1 + \frac{m_2}{2} l_1\right) g \sin\varphi \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (2.9)$$

The non-conservative generalized force is given by

$$\boldsymbol{\tau} = \begin{bmatrix} M_\varphi & 0 & 0 & 0 \end{bmatrix}^T \qquad (2.10)$$

Finally, the differential–algebraic equations describing the motion of the mechanism are written as

$$\boldsymbol{M}(\boldsymbol{s})\ddot{\boldsymbol{s}} + \boldsymbol{C}(\boldsymbol{s},\dot{\boldsymbol{s}})\dot{\boldsymbol{s}} + \boldsymbol{g}(\boldsymbol{s}) + \mathrm{d}(\boldsymbol{s},\dot{\boldsymbol{s}}) = \boldsymbol{t} - \boldsymbol{F}_s^T \boldsymbol{l} \quad \mathbf{f}(\mathbf{s}) = \mathbf{0} \qquad (2.11)$$

2.2 Dynamic Simulation of the Mechanism

To perform the numerical simulation, the mechanism parameters are chosen as follows:

$$h = 0.12m; l_1 = 0.08m; l_2 = 0.3m; \mathrm{d} = 0.1\mathrm{m}; \alpha = 0\,\mathrm{rad};$$

$$m_1 = 0.12kg,\ m_2 = 0.5kg,\ m_3 = 2kg,\ J_1 = 0.0001kgm^2,$$
$$g = 9.81m/s^2$$

The driving couple is assumed to have the following torque:

$$\tau_\varphi = \tau_0 \sin\Omega t = 0.005\sin(2\pi t)\left[Nm\right]$$

The initial conditions are selected as follows:

$$\varphi(0) = 0(\mathrm{rad}); \beta = 0.841069(\mathrm{rad}); \gamma = 0.201082(\mathrm{rad}); \mathrm{x}_C = 2(\mathrm{m});$$
$$\dot{\varphi}(0) = 2 * \pi(\mathrm{rad/s});$$

III. RESULTS AND DISCUSSIONS

Numerical simulations were performed using MATLAB based on the algorithms presented in the previous section. Selected results are presented in the plots shown in Figures 3.1–3.4.

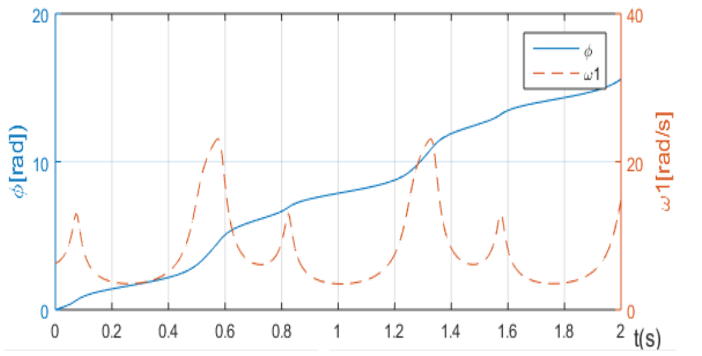

Figure 3.1: Graph of $\varphi, \dot{\varphi}$

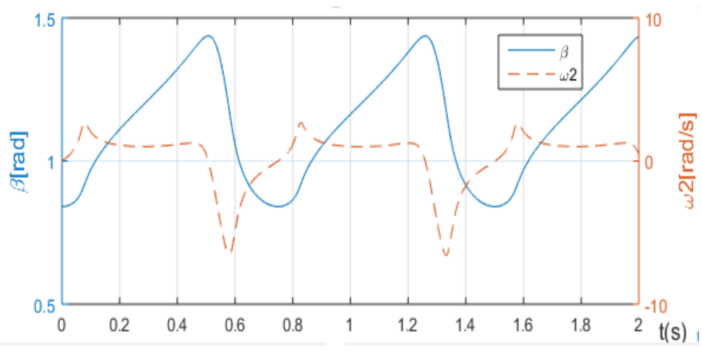

Figure 3.2: Graph of $\beta, \dot{\beta}$

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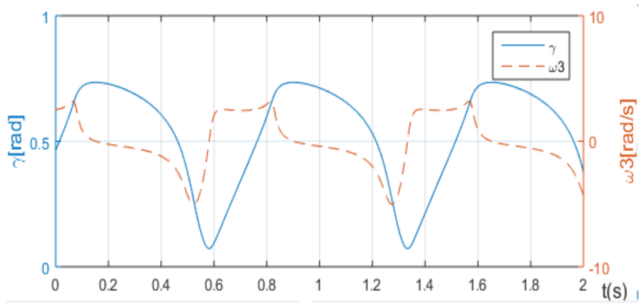

Figure 3.3: Graph of $\gamma, \dot{\gamma}$

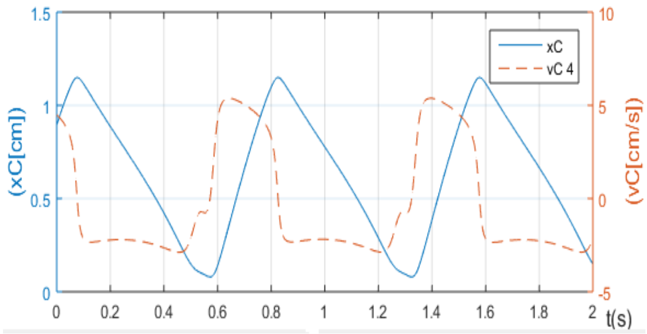

Figure 3.4: Graph of $x_C, \dot{x}_C$

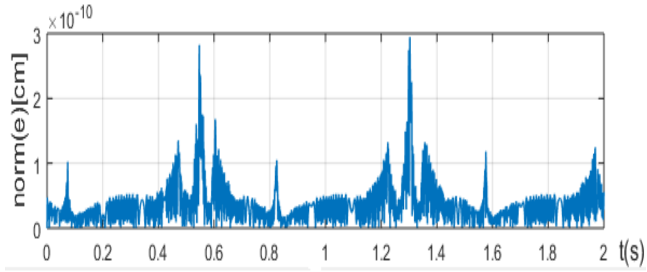

Figure 3.5: Graph of Constraint equation error

IV. CONCLUSION

In this paper, a method is proposed to solve the inverse dynamics problem. Instead of using the conventional approach, which requires solving algebraic equations, the authors suggest transforming the problem into solving second-order differential equations. In this formulation, the kinematic requirements of the end-effector are treated as program constraints, which can be considered an extension of the concept of first integrals [2, 3, 9]. The paper addresses systems subjected to program constraints. However, the proposed idea can also be applied to systems with physical constraints involving non-ideal constraint forces.

The obtained results are illustrated through simulations performed in Matlab for the motion of a two-degree-of-freedom excavator manipulator.

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ISSN (online): 2581-3048
Volume 10, Issue 6, pp 55-59, June-2026
https://doi.org/10.47001/IRJIET/2026.106005

Citation of this Article:

Mai-Anh Phạm. (2026). Forward Dynamic Analysis of a Spatial Four-Bar Linkage. *International Research Journal of Innovations in Engineering and Technology - IRJIET*, 10(6), 55-59. Article DOI https://doi.org/10.47001/IRJIET/2026.106005

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