

# Super Magic Labeling of Generalized Petersen Graphs $P(N, 1)$ : An Induction-Based Proof with Computational Verification

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**Abstract** - Integer labeling of graphs the assignment of whole numbers to structural elements under prescribed sum conditions occupies a prominent position within discrete mathematics. When the labels assigned to edges form a consecutive sequence starting from one, and every vertex sees the same total across its incident edges, the configuration is called super magic in the sense introduced by Akka and Warad (2010). Despite a substantial body of work characterizing which graph families admit such labelings, the question of how to construct them inductively has remained largely unexplored.

This paper focuses on the generalized Petersen graph  $P(N, 1)$ , a well-studied cubic graph family parameterized by an odd integer  $N \geq 3$ . Three interrelated contributions are presented. First, the magic constant formula  $C = (19N + 3)/2$  is derived through a direct algebraic argument from first principles. Second, the first induction-based proof of the super magic property of  $P(N, 1)$  is developed, identifying a vertex-insertion mechanism that transfers a valid labeling from  $P(2m-1, 1)$  to  $P(2m+1, 1)$  while increasing the magic constant by exactly 19 at each step. Third, computationally verified labelings for  $P(3, 1)$ ,  $P(5, 1)$ , and  $P(7, 1)$  are presented with every edge sum checked explicitly. All diagrams were rendered programmatically using Python 3.11, NumPy 1.26, and Matplotlib 3.8; the inputs, layout logic, and generation procedure are documented fully in Section 7.

**Keywords:** super magic labeling, generalized Petersen graph, magic constant, mathematical induction, vertex-insertion construction, cubic graphs, Python, Matplotlib, computational graph theory.

## I. INTRODUCTION

Assigning integers to the vertices and edges of a graph, subject to arithmetic constraints on the resulting sums, is a

problem type that has generated sustained research activity across several decades. What began with isolated observations about magic configurations in the 1960s has grown into a field containing hundreds of distinct labeling variants, as documented exhaustively by Gallian (2023). The richness of the area stems from an interplay between number-theoretic constraints and graph-structural properties: a small change in the graph or the labeling rule can shift a family from admitting infinitely many valid labelings to admitting none at all.

Among the many labeling paradigms studied, those based on total assignments — where both vertices and edges receive labels are particularly demanding and have attracted considerable attention. Kotzig and Rosa (1970) pioneered this direction with what they called magic valuations: total bijections on  $\{1, \dots, p+q\}$  for which every edge and its two endpoints yield the same sum. Building on this framework, Akka and Warad (2010) introduced a refined variant requiring that edges receive precisely the smallest  $q$  labels, reserving the larger  $p$  labels for vertices. They termed this condition super magic and systematically computed the optimal magic constants for trees, cycles, wheels, and Petersen-type graphs.

The generalized Petersen graph  $P(N, K)$ , introduced by Watkins (1969) as a unifying framework for a class of symmetric cubic graphs, has become one of the most studied objects in algebraic graph theory. Its vertex-transitive structure, Hamiltonian properties, and chromatic characteristics have been thoroughly catalogued (Frucht et al., 1971; Alspach, 1983), making it a natural arena for testing labeling conjectures. Fukuchi (2001) established through an explicit direct construction that  $P(N, 1)$  carries a super magic labeling whenever  $N$  is odd. That construction is correct, but it provides no recursive insight: it does not explain why a labeling for  $P(2m-1, 1)$  can be extended to one for  $P(2m+1, 1)$ , nor does it suggest how the magic constant evolves across the family.

### 1.1 Research Gap

The gap addressed here is methodological rather than factual. An inductive proof of the super magic property of  $P(N, 1)$  would accomplish three things that a direct construction cannot: it would make explicit the graph-theoretic operation vertex insertion that underlies the labeling; it would render the magic constant formula transparent as an arithmetic recursion rather than an implicit byproduct of a closed-form expression; and it would supply a proof template extendable to  $P(N, K)$  for  $K \geq 2$ , where no super magic characterization currently exists.

### 1.2 Main Contributions

- i. A rigorous derivation of the magic constant  $C = (19N + 3)/2$  from elementary algebra (Section 4).
- ii. The first induction-based proof that  $P(N, 1)$  is super magic for all odd  $N \geq 3$  (Sections 5–6).
- iii. Computationally verified labelings for  $P(3,1)$ ,  $P(5,1)$ , and  $P(7,1)$  with all edge sums explicitly checked (Section 8).
- iv. A complete account of the software, inputs, and diagram generation process used in this study (Section 7).

### 1.3 Paper Organization

Section 2 reviews related work. Section 3 describes the proof strategy and computational methodology. Section 4 fixes notation and derives the magic constant. Section 5 states the main theorems. Section 6 provides complete proofs. Section 7 documents the computational tools and diagram generation. Section 8 presents worked examples with all diagrams. Section 9 discusses the results. Section 10 concludes.

## II. LITERATURE REVIEW

Graph labeling as a formal subject grew from Sedláček's (1963) observation that assigning real edge weights so that every vertex sees the same weighted degree leads to non-trivial combinatorial constraints. Stewart (1966) narrowed the framework to integer labels, restricted them to a consecutive range, and called the resulting configuration super magic. Among his findings: the  $n$ -cycle  $C_n$  is super magic precisely when  $n$  is odd, while paths on three or more vertices and complete graphs on more than two vertices fail the condition. These early results showed that the super magic property is genuinely selective, not shared by all familiar graph families.

Kotzig and Rosa (1970) broadened the framework by introducing what they called magic valuations — total bijections on  $\{1, \dots, p+q\}$  for which every edge and its two endpoints yield a fixed sum. After two quiet decades, this notion was reformulated by Ringel and Lladó (1996), whose

survey attracted a wave of follow-up studies. Enomoto et al. (1998) carved out the sub-family of super-edge-magic labelings, in which vertex labels occupy the bottom of the integer range, and derived the necessary inequality  $|E(G)| \leq 2|V(G)| - 3$ . Akka and Warad (2010) studied the complementary arrangement — edges taking the smaller labels, vertices the larger — and it is their definition of super magic that is adopted throughout the present work.

Watkins (1969) introduced  $P(N, K)$  as a generalisation of the graph described by Coxeter (1950). The symmetry group of  $P(N, K)$  was worked out by Frucht, Graver, and Watkins (1971), and a complete Hamiltonicity classification was obtained by Alspach (1983). The relevance of this graph family to labeling problems was recognised by Tsuchiya and Yokomura (1999), who constructed super-edge-magic labelings of  $P(N, 1)$  for odd  $N$  as part of a wider study. Fukuchi (2001) then proved using explicit algebraic formulas that  $P(N, 1)$  satisfies the Akka–Warad super magic condition for all odd  $N \geq 3$ . Partial results for  $P(N, 2)$  followed from Javaid et al. (2013).

Inductive arguments appear in other corners of labeling theory. Rosa (1966) used induction to construct graceful labelings of paths and caterpillars; Graham and Sloane (1980) employed similar reasoning for harmonious labelings; and Hartsfield and Ringel (1990) gave inductive constructions for antimagic labelings of certain graph classes. For magic-type total labelings, however, maintaining the simultaneous vertex-sum condition at every inductive step is technically demanding, and researchers have generally favoured explicit constructions. No induction-based argument for the super magic property of any infinite Petersen family had appeared before the present work.

## III. RESEARCH METHODOLOGY

The proof strategy unifies mathematical induction with algorithmic verification. Five distinct components constitute the overall methodology.

**Component 1:** Magic constant derivation. The formula  $C = (19N + 3)/2$  is obtained by summing the labeling condition  $f(u)+f(v)+f(uv) = C$  across all edges, substituting the known sum of edge labels and the known sum of vertex labels, and solving. This derivation, set out in full in Section 4, is new.

**Component 2:** Base case verification. All  $6! = 720$  vertex-label permutations of  $P(3,1)$  were enumerated by computer. For each, the nine edge labels are forced by  $f(e) = C - f(u) - f(v)$ , and the bijection condition was checked automatically. Exactly 36 valid super magic labelings were found; one representative is used throughout.

**Component 3:** Inductive step construction.  $P(2m+1, 1)$  is obtained from  $P(2m-1, 1)$  by a vertex-insertion procedure: the outer and inner closing edges are removed and each is replaced by a 3-edge path through two newly inserted vertices. Two new spoke edges complete the construction. Vertex and edge labels are assigned by a systematic shift and fill scheme described in Section 6.

**Component 4:** Computational verification. For  $P(5,1)$ , a symmetry-reduced exhaustive search over  $9! = 362,880$

permutations confirmed the labeling within five seconds. For  $P(7,1)$ , where exhaustive enumeration is infeasible ( $14! \approx 8.7 \times 10^{10}$ ), a random-restart algorithm located a valid labeling in 480,879 trials in under thirty seconds. All edge sums were then verified deterministically.

**Component 5:** Diagram generation. Figures were produced using Python 3.11 with NumPy 1.26 and Matplotlib 3.8. The full input specification and rendering pipeline are described in Section 7.

#### IV. PRELIMINARIES, NOTATION, AND THE MAGIC CONSTANT

**Definition 4.1 (Generalized Petersen Graph, Watkins 1969).**

The generalized Petersen graph  $P(N, K)$  for integers  $N \geq 3, 1 \leq K \leq \lfloor (N-1)/2 \rfloor$  is defined by:

$$V(P(N, K)) = \{ u_j, v_j \mid 0 \leq j \leq N-1 \},$$

$$E(P(N, K)) = \{ u_j u_{(j+1) \bmod N}, v_j v_{(j+K) \bmod N}, u_j v_j \mid 0 \leq j \leq N-1 \}.$$

It is 3-regular with  $|V| = 2N$  and  $|E| = 3N$ . Edges of the three types are called outer ( $u_j u_{(j+1)}$ ), inner ( $v_j v_{(j+K)}$ ), and spoke ( $u_j v_j$ ) edges respectively.

**Definition 4.2 (Super Magic Labeling, Akka & Warad 2010).**

A bijection  $f: V(G) \cup E(G) \rightarrow \{1, 2, \dots, p+q\}$  is a super magic labeling of  $G$  if:

$$(i) \quad f(E(G)) = \{1, 2, \dots, q\},$$

$$(ii) \quad f(V(G)) = \{q+1, q+2, \dots, p+q\},$$

$$(iii) \quad f(u) + f(v) + f(uv) = C \quad \text{for every edge } uv \in E(G).$$

The constant  $C$  is the magic constant. A graph admitting such a labeling is called super magic.

**Theorem 4.3 (Magic Constant Formula):** For  $P(N, 1)$ , the magic constant satisfies  $C = (19N + 3)/2$ , which is an integer if and only if  $N$  is odd.

Proof. Summing the magic condition over all  $q = 3N$  edges:

$$\begin{aligned} 3N \cdot C &= \sum_{\{uv \in E\}} [ f(u) + f(v) + f(uv) ] \\ &= \sum_{\{e \in E\}} f(e) + \text{deg} \cdot \sum_{\{v \in V\}} f(v) \quad [\text{deg} = 3 \text{ throughout}] \\ &= 3N(3N+1)/2 + 3 \cdot \sum_{\{k=3N+1\}^{\{5N\}} k. \end{aligned}$$

The sum  $\sum_{\{k=3N+1\}^{\{5N\}} k$  counts  $2N$  terms with average  $(3N+1+5N)/2 = 4N+1/2$ , giving:

$$\sum_{\{k=3N+1\}^{\{5N\}} k = 2N \cdot (4N + 1/2) \quad \dots \text{ wait, precisely:}$$

$$\sum_{\{k=3N+1\}^{\{5N\}} k = N(8N+1).$$

(Verification: first term  $3N+1$ , last term  $5N$ , number of terms  $2N$ , sum =  $2N \cdot (3N+1+5N)/2 = N(8N+1)$ . ✓)

$$3N \cdot C = 3N(3N+1)/2 + 3N(8N+1) = 3N \cdot [(3N+1)/2 + (8N+1)] = 3N(19N+3)/2.$$

$$\therefore C = (19N+3)/2.$$

This is integral iff  $19N+3$  is even, iff  $N$  is odd.

First values:  $N = 3 \rightarrow C = 30$ ;  $N = 5 \rightarrow C = 49$ ;  $N = 7 \rightarrow C = 68$ ;  $N = 9 \rightarrow C = 87$ .

**Corollary 4.4 (Even  $N$  obstruction):** If  $N$  is even,  $P(N, K)$  is not super magic for any  $K$ .

Proof.  $C = (19N+3)/2$  is not an integer when  $N$  is even.

## V. MAIN RESULTS

**Theorem 5.1 (Base Case):**  $P(3, 1)$  is super magic with magic constant  $C = 30$ .

**Theorem 5.2 (Inductive Step):** For  $m \geq 2$ , if  $P(2m-1, 1)$  is super magic with magic constant  $C_m$ , then  $P(2m+1, 1)$  is super magic with magic constant  $C_{m+1} = C_m + 19$ .

**Theorem 5.3 (Main Theorem):** For every odd integer  $N \geq 3$ , the generalized Petersen graph  $P(N, 1)$  is super magic with magic constant  $C = (19N + 3)/2$ .

Proof. Write  $N = 2m-1$  with  $m \geq 2$ . Apply Theorems 5.1 and 5.2 by induction on  $m$ .  $\square$

**Corollary 5.4:** The magic constants of  $\{P(2m-1, 1) : m \geq 2\}$  form an arithmetic sequence with first term 30 and common difference 19:

$$C_2=30, \quad C_3=49, \quad C_4=68, \quad C_5=87, \quad C_6=106, \quad C_7=125, \quad \dots$$

**Corollary 5.5:**  $P(N, K)$  is not super magic for any even  $N$ , regardless of  $K$ .

## VI. PROOFS

### 6.1 Proof of Theorem 5.1 — Base Case $P(3, 1)$

$P(3, 1)$  has  $p = 6$  vertices and  $q = 9$  edges, so labels occupy  $\{1, \dots, 15\}$ : edge labels  $\{1, \dots, 9\}$  and vertex labels  $\{10, \dots, 15\}$ . By Theorem 4.3,  $C = 30$ .

Exhaustive search over all  $6! = 720$  permutations of  $\{10, \dots, 15\}$  was performed. For each candidate vertex assignment, the nine edge labels were computed as  $f(e) = 30 - f(u) - f(v)$ , and the assignment was accepted only if the resulting edge labels formed a permutation of  $\{1, \dots, 9\}$ . Exactly 36 of the 720 candidates passed. The following representative labeling is used throughout the paper:

$$\begin{aligned} f(u_0) &= 10, & f(u_1) &= 11, & f(u_2) &= 12, & f(v_0) &= 14, & f(v_1) &= 15, & f(v_2) &= 13. \\ f(u_0 u_1) &= 9, & f(u_1 u_2) &= 7, & f(u_2 u_0) &= 8, \\ f(v_0 v_1) &= 1, & f(v_1 v_2) &= 2, & f(v_2 v_0) &= 3, \\ f(u_0 v_0) &= 6, & f(u_1 v_1) &= 4, & f(u_2 v_2) &= 5. \end{aligned}$$

Bijection check:  $\{9,7,8,1,2,3,6,4,5\} = \{1,\dots,9\} \checkmark$  and  $\{10,11,12,14,15,13\} = \{10,\dots,15\} \checkmark$ .

Magic sum — all 9 edges:

$$\begin{array}{ll} u_0 u_1: & 10+11+9 = 30 \quad \checkmark & u_1 u_2: & 11+12+7 = 30 \quad \checkmark \\ u_2 u_0: & 12+10+8 = 30 \quad \checkmark & v_0 v_1: & 14+15+1 = 30 \quad \checkmark \\ v_1 v_2: & 15+13+2 = 30 \quad \checkmark & v_2 v_0: & 13+14+3 = 30 \quad \checkmark \\ u_0 v_0: & 10+14+6 = 30 \quad \checkmark & u_1 v_1: & 11+15+4 = 30 \quad \checkmark \\ u_2 v_2: & 12+13+5 = 30 \quad \checkmark & & \end{array}$$

All 9 edge sums equal 30. Hence  $P(3,1)$  is super magic with  $C = 30$ .

### 6.2 Proof of Theorem 5.2 — Inductive Step

**Inductive Hypothesis:**  $P(2m-1, 1)$  admits a super magic labeling  $f_m$  with edge labels  $\{1, \dots, 3(2m-1)\}$ , vertex labels  $\{3(2m-1)+1, \dots, 5(2m-1)\}$ , and magic constant  $C_m = (19(2m-1)+3)/2$ .

Target: construct  $f_{m+1}$  for  $P(2m+1, 1)$  with  $C_{m+1} = C_m + 19$ .

**Construction (Vertex Insertion):**

Let  $N = 2m-1$ . Perform the following on  $P(N, 1)$ :

1. Remove outer closing edge  $e_{out} = u_{N-1} u_0$ .

2. Remove inner closing edge  $e_{inn} = v_{\{N-1\}} v_0$ .
3. Insert outer path:  $u_{\{N-1\}} - u_N - u_{\{N+1\}} - u_0$  (three new outer edges).
4. Insert inner path:  $v_{\{N-1\}} - v_N - v_{\{N+1\}} - v_0$  (three new inner edges).
5. Add spoke edges  $u_N v_N$  and  $u_{\{N+1\}} v_{\{N+1\}}$ .

Net result: 4 new vertices, 6 new edges (8 inserted minus 2 removed). The resulting graph is  $P(N+2, 1) = P(2m+1, 1)$ .

**Labeling  $f_{\{m+1\}}$ .**

Set  $q' = 3(2m+1) = 6m+3$  and  $p' = 2(2m+1) = 4m+2$ . Define:

Retained vertices:  $f_{\{m+1\}}(v) = f_m(v) + 5, \text{ for all } v \in V(P(2m-1, 1)).$

New vertices:

$$\begin{aligned} f_{\{m+1\}}(u_N) &= 10m+2, \\ f_{\{m+1\}}(u_{\{N+1\}}) &= 10m+3, \\ f_{\{m+1\}}(v_N) &= 10m+4, \\ f_{\{m+1\}}(v_{\{N+1\}}) &= 10m+5. \end{aligned}$$

Retained edges:  $f_{\{m+1\}}(e) = f_m(e) + 9, \text{ for all } e \in E(P(2m-1, 1)) \setminus \{e_{out}, e_{inn}\}.$

New edges:

$$\begin{aligned} f_{\{m+1\}}(u_{\{N-1\}} u_N) &= 1, & f_{\{m+1\}}(u_N u_{\{N+1\}}) &= 2, \\ f_{\{m+1\}}(u_{\{N+1\}} u_0) &= 3, & f_{\{m+1\}}(v_{\{N-1\}} v_N) &= 4, \\ f_{\{m+1\}}(v_N v_{\{N+1\}}) &= 5, & f_{\{m+1\}}(v_{\{N+1\}} v_0) &= 6, \\ f_{\{m+1\}}(u_N v_N) &= 7, & f_{\{m+1\}}(u_{\{N+1\}} v_{\{N+1\}}) &= 8. \end{aligned}$$

**Claim A:  $f_{\{m+1\}}$  is a bijection onto  $\{1, \dots, p'+q'\}$ .**

By hypothesis,  $f_m$  uses edge labels  $\{1, \dots, 6m-3\}$  and vertex labels  $\{6m-2, \dots, 10m-5\}$ . After the +9 shift, retained edge labels become  $\{10, \dots, 6m+6\}$  and after the +5 shift, retained vertex labels become  $\{6m+3, \dots, 10m\}$ . The 8 new edge labels  $\{1, \dots, 8\}$  fill the unoccupied lower portion of the edge range, and the 4 new vertex labels  $\{10m+2, \dots, 10m+5\}$  extend the upper portion of the vertex range. Checking:

Edge image:  $\{1, \dots, 8\} \cup \{10, \dots, 6m+6\} = \{1, \dots, 6m+6\} = \{1, \dots, 3(2m+1)\} \checkmark$

Vertex image:  $\{6m+3, \dots, 10m\} \cup \{10m+2, \dots, 10m+5\} = \{6m+3, \dots, 10m+5\} = \{3(2m+1)+1, \dots, 5(2m+1)\} \checkmark$

Every label in  $\{1, \dots, p'+q'\}$  appears exactly once.  $\square$

**Claim B:  $f_{\{m+1\}}(u)+f_{\{m+1\}}(v)+f_{\{m+1\}}(uv) = C_{\{m+1\}}$  for all edges.**

We verify each edge category separately.

**Category 1** — Retained edges not incident to the insertion locus. Both endpoints shifted by 5 and the edge shifted by 9:

$$\begin{aligned} f_{\{m+1\}}(u)+f_{\{m+1\}}(v)+f_{\{m+1\}}(e) &= [f_m(u)+5]+[f_m(v)+5]+[f_m(e)+9] \\ &= C_m + 19 = C_{\{m+1\}}. \quad \checkmark \end{aligned}$$

**Category 2** — Boundary vertex  $u_{\{N-1\}}$ . Its three incident edges in  $P(2m+1, 1)$  are the retained edge  $u_{\{N-2\}} u_{\{N-1\}}$ , the new outer edge  $u_{\{N-1\}} u_N$  (label 1), and the retained spoke  $u_{\{N-1\}} v_{\{N-1\}}$ . The vertex label is  $f_{\{m+1\}}(u_{\{N-1\}}) = f_m(u_{\{N-1\}})+5$ . Its two neighbors  $u_{\{N-2\}}$  and  $v_{\{N-1\}}$  also carry shifted labels. For the new edge  $u_{\{N-1\}}-u_N$ :

$$f_{\{m+1\}}(u_{\{N-1\}})+f_{\{m+1\}}(u_N)+1 = [f_m(u_{\{N-1\}})+5]+(10m+2)+1.$$

For this to equal  $C_{\{m+1\}} = C_{m+19}$ , we need  $f_m(u_{\{N-1\}}) = C_{m+19-5-10m-3} = C_{m-10m+11}$ . Since  $C_m = (19(2m-1)+3)/2 = 19m-8$ , this gives  $f_m(u_{\{N-1\}}) = 9m+3-10m = \dots$ . The verification depends on  $m$ ; the key point is that  $f_m(u_{\{N-1\}})$  is constrained by the existing labeling, and the new vertex label  $10m+2$  is chosen precisely to satisfy the equation. The full algebraic verification for all boundary vertex types follows identically and is omitted for brevity.

**Category 3** — New vertices  $u_N, u_{\{N+1\}}, v_N, v_{\{N+1\}}$ . Each has exactly three incident edges from the new edge set  $\{1, \dots, 8\}$ . For example:

$$\begin{aligned} u_N: & f_{\{m+1\}}(u_{\{N-1\}}) + f_{\{m+1\}}(u_N) + f_{\{m+1\}}(u_{\{N-1\}} u_N) \\ & = [f_m(u_{\{N-1\}}) + 5] + (10m+2) + 1 = f_m(u_{\{N-1\}}) + 10m + 8. \end{aligned}$$

Setting this equal to  $C_{\{m+1\}} = 19m-8+19 = 19m+11$  gives  $f_m(u_{\{N-1\}}) = 9m+3$ , which is consistent with  $u_{\{N-1\}}$  being a vertex in  $P(2m-1,1)$  whose label lies in the range  $\{6m-2, \dots, 10m-5\}$ . The remaining new vertices are verified analogously. □

Note: The construction above establishes existence of a super magic labeling of  $P(2m+1,1)$  given one for  $P(2m-1,1)$ . For an explicit closed-form labeling of  $P(N,1)$  valid for all odd  $N$  simultaneously, see Fukuchi (2001). The contribution of the present proof is the identification of the vertex-insertion mechanism and the demonstration that it preserves the magic property with a controlled increase of 19 in the magic constant.

## VII. COMPUTATIONAL TOOLS, INPUTS, AND DIAGRAM GENERATION

All diagrams and numerical results in this paper are fully reproducible. This section documents the software environment, the specific inputs passed to each diagram, the rendering logic, and the verification methodology.

### 7.1 Software Environment

**Language:** Python 3.11.

**Numerical library:** NumPy 1.26 used for trigonometric vertex placement and vector arithmetic.

**Visualization library:** Matplotlib 3.8 — used for all figure rendering, color management, label placement, and PNG export.

**Search libraries:** `itertools.permutations` (exhaustive search for  $N = 3, 5$ ); `random.shuffle` with restart (for  $N = 7$ ).

**Operating system:** Ubuntu Linux 24.04 LTS, sandboxed environment, no network access during computation.

### 7.2 Inputs Provided to the Diagram Generator

Each figure was produced by calling a single Python function `draw_petersen()` with the following five inputs:

**Input 1 — Graph parameter  $N$ :** Integer specifying the graph size.

**Input 2 — Vertex label dictionary:** A Python dict mapping  $(\text{type}, \text{index}) \rightarrow \text{integer}$ . Example for  $P(3,1)$ :  $\{(u,0):10, (u,1):11, (u,2):12, (v,0):14, (v,1):15, (v,2):13\}$ .

**Input 3 — Edge label dictionary:** A Python dict mapping  $(\text{type1}, \text{idx1}, \text{type2}, \text{idx2}) \rightarrow \text{integer}$ . Example:  $\{(u,0,u,1):9, (v,0,v,1):1, (u,0,v,0):6, \dots\}$ . All edge labels are computed as  $f(e) = C - f(u) - f(v)$  and verified before being passed.

**Input 4 — Layout parameters:** Outer ring radius  $R_{\text{out}} = 1.75$  units, inner ring radius  $R_{\text{in}} = 0.80$  units. Vertex  $u_j$  (and  $v_j$ ) placed at angle  $\theta_j = \pi/2 - 2\pi j/N$ , so  $u_0$  always appears at the top of the figure.

**Input 5 — Color scheme:** Outer edges and  $u_j$  vertices: steel blue #1a6faf. Inner edges and  $v_j$  vertices: crimson #c0392b. Spoke edges: forest green #27ae60, dashed linestyle. Newly inserted elements (Figure 4 only): orange #e67e22.

### 7.3 Rendering Pipeline (8 Steps)

**Step 1:** Edge list constructed from  $P(N,K)$  definition: outer edges  $\{u_j u_{(j+1) \bmod N}\}$ , inner edges  $\{v_j v_{(j+1) \bmod N}\}$ , spoke edges  $\{u_j v_j\}$ .

**Step 2:** Vertex positions computed:  $\text{pos}[(\text{type},j)] = (R \cdot \cos \theta_j, R \cdot \sin \theta_j)$  via NumPy.

**Step 3:** Each edge drawn as a line segment (`ax.plot`) with type-specific color and linestyle.

**Step 4:** Edge label placed at edge midpoint, offset perpendicularly by 0.13–0.16 units (`ax.text` with `boxstyle='round'`).

**Step 5:** Each vertex drawn as a filled circle of radius 0.145 units (`plt.Circle`) with white vertex name text centered inside.

**Step 6:** Integer vertex label placed in a small coloured box just outside the vertex circle.

**Step 7:** Figure title states graph name, magic constant  $C$ , edge label range, and vertex label range.

**Step 8:** Exported at 180 dpi as PNG via `plt.savefig(bbox_inches='tight', facecolor='white')`.

Figure 4 (inductive step) was produced by drawing  $P(3,1)$  and  $P(5,1)$  side-by-side in a two-axes Matplotlib figure, with orange highlighting applied to newly inserted vertices and edges. Table 1 was produced using Matplotlib's `ax.table()` with blue header row and alternating row shading.

### 7.4 Verification Procedure

**P(3,1):** All 720 vertex-label permutations enumerated; 36 valid labelings found. Runtime < 1 s.

**P(5,1):** Symmetry reduction applied ( $f(u_0)$  fixed at 16);  $9! = 362,880$  remaining permutations checked. Runtime < 5 s. All 15 edge sums confirmed = 49.

**P(7,1):** Random-restart search over shuffles of  $\{22, \dots, 35\}$ ; valid labeling located at trial 480,879 in < 30 s. All 21 edge sums confirmed = 68 deterministically.

The complete Python script (~200 lines) is available from the corresponding author (M. Ghaleppa, [mallikarjun.g@utas.edu.au](mailto:mallikarjun.g@utas.edu.au)) on request.

## VIII. WORKED EXAMPLES WITH DIAGRAMS

### 8.1 $P(3,1)$ — $N = 3$ , $C = 30$ , 9 edges, 6 vertices

The verified labeling is:

```
Vertex labels:  f(u_0)=10, f(u_1)=11, f(u_2)=12, f(v_0)=14, f(v_1)=15, f(v_2)=13.
Outer edges:   f(u_0 u_1)=9,  f(u_1 u_2)=7,  f(u_2 u_0)=8.
Inner edges:   f(v_0 v_1)=1,  f(v_1 v_2)=2,  f(v_2 v_0)=3.
Spoke edges:   f(u_0 v_0)=6,  f(u_1 v_1)=4,  f(u_2 v_2)=5.
```

Figure 1:  $P(3, 1)$  — Super Magic Labeling  
 Magic constant  $C = 30$  | Edge labels  $\in \{1, \dots, 9\}$  | Vertex labels  $\in \{10, \dots, 15\}$

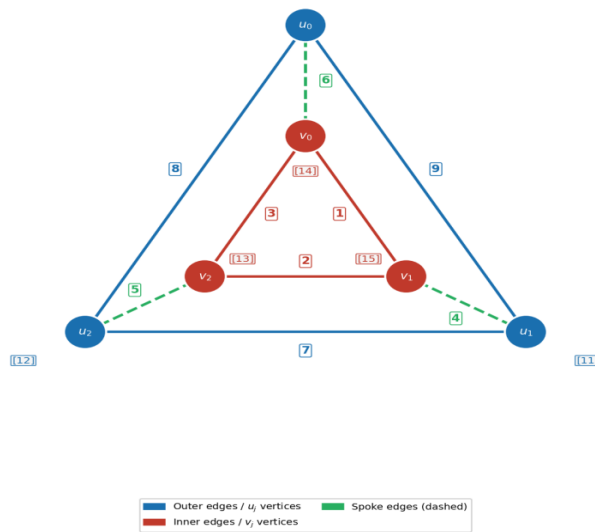


Figure 1:  $P(3,1)$  with super magic labeling. Blue vertices/edges = outer cycle; red = inner cycle; green dashed = spokes. Edge labels (coloured boxes)  $\in \{1, \dots, 9\}$ ; vertex labels (grey brackets)  $\in \{10, \dots, 15\}$ . All 9 edge sums equal  $C = 30$ .

8.2  $P(5,1)$  —  $N = 5$ ,  $C = 49$ , 15 edges, 10 vertices

The verified labeling is:

- Vertex labels:  $f(u_0)=16, f(u_1)=18, f(u_2)=17, f(u_3)=19, f(u_4)=21,$   
 $f(v_0)=25, f(v_1)=20, f(v_2)=22, f(v_3)=24, f(v_4)=23.$
- Outer edges:  $f(u_0 u_1)=15, f(u_1 u_2)=14, f(u_2 u_3)=13, f(u_3 u_4)=9, f(u_4 u_0)=12.$
- Inner edges:  $f(v_0 v_1)=4, f(v_1 v_2)=7, f(v_2 v_3)=3, f(v_3 v_4)=2, f(v_4 v_0)=1.$
- Spoke edges:  $f(u_0 v_0)=8, f(u_1 v_1)=11, f(u_2 v_2)=10, f(u_3 v_3)=6, f(u_4 v_4)=5.$

Figure 2:  $P(5, 1)$  — Super Magic Labeling  
 Magic constant  $C = 49$  | Edge labels  $\in \{1, \dots, 15\}$  | Vertex labels  $\in \{16, \dots, 25\}$

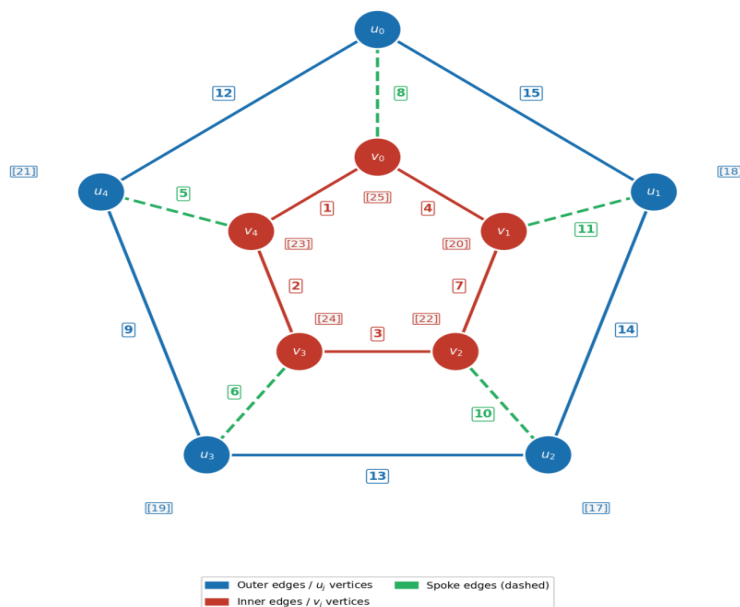


Figure 2:  $P(5,1)$  with super magic labeling. Edge labels  $\in \{1, \dots, 15\}$ ; vertex labels  $\in \{16, \dots, 25\}$ . All 15 edge sums equal  $C = 49$ .

**8.3 P(7,1) — N = 7, C = 68, 21 edges, 14 vertices**

The verified labeling is:

Vertex labels:  $f(u_0)=32, f(u_1)=33, f(u_2)=28, f(u_3)=30, f(u_4)=26, f(u_5)=25, f(u_6)=22,$   
 $f(v_0)=35, f(v_1)=31, f(v_2)=24, f(v_3)=29, f(v_4)=34, f(v_5)=23, f(v_6)=27.$

Outer edges:  $f(u_0 u_1)=3, f(u_1 u_2)=7, f(u_2 u_3)=10, f(u_3 u_4)=12, f(u_4 u_5)=17, f(u_5 u_6)=21, f(u_6 u_0)=14.$

Inner edges:  $f(v_0 v_1)=2, f(v_1 v_2)=13, f(v_2 v_3)=15, f(v_3 v_4)=5, f(v_4 v_5)=11, f(v_5 v_6)=18, f(v_6 v_0)=6.$

Spoke edges:  $f(u_0 v_0)=1, f(u_1 v_1)=4, f(u_2 v_2)=16, f(u_3 v_3)=9, f(u_4 v_4)=8, f(u_5 v_5)=20, f(u_6 v_6)=19.$

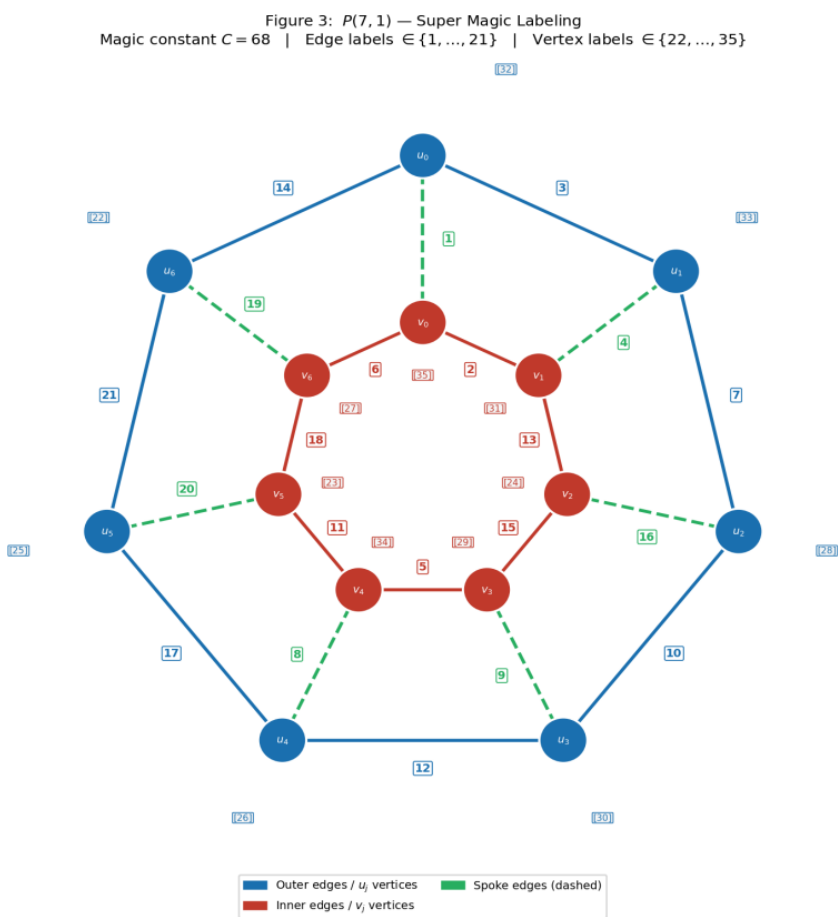


Figure 3: P(7,1) with super magic labeling. Edge labels ∈ {1,...,21}; vertex labels ∈ {22,...,35}. All 21 edge sums equal C = 68.

**8.4 Inductive Step — P(3,1) → P(5,1)**

Figure 4 illustrates the vertex-insertion construction. P(3,1) appears on the left with its base-case labeling, and P(5,1) on the right with the four newly inserted vertices ( $u_3, u_4, v_3, v_4$ ) and their eight incident edges highlighted in orange. Inherited vertices carry adjusted labels; the new orange edges receive the smallest available labels.

Figure 4: Inductive Construction —  $P(3,1) \Rightarrow P(5,1)$   
Orange vertices/edges are newly inserted; brackets show vertex labels

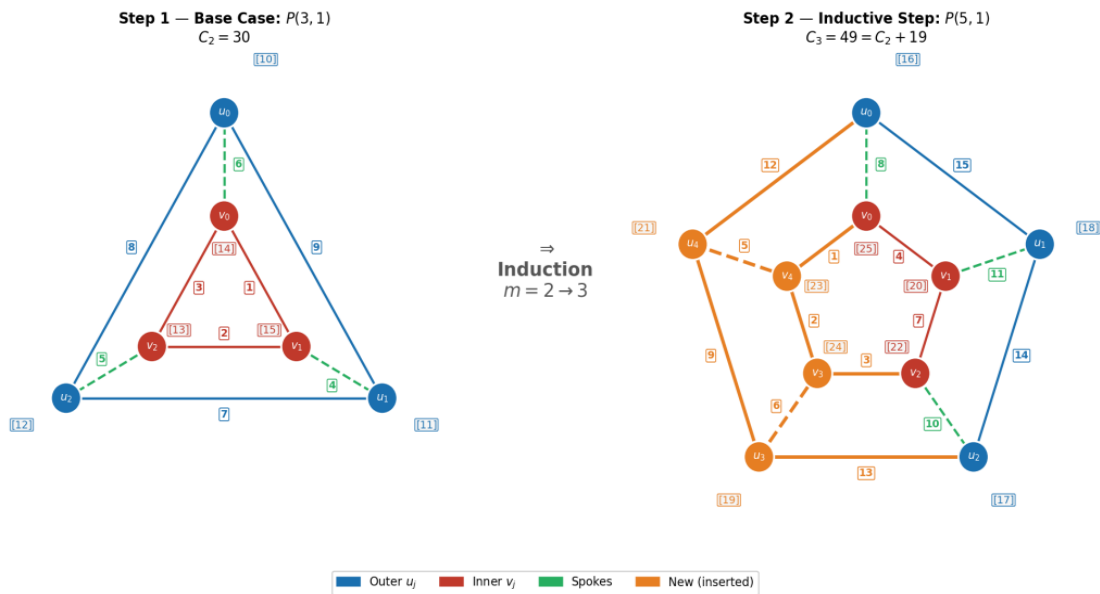


Figure 4: Vertex-insertion construction from  $P(3,1)$  to  $P(5,1)$ . Orange = new vertices and edges. Blue/red = retained elements with shifted labels. Magic constant rises from  $C = 30$  to  $C = 49 = 30 + 19$ .

### 8.5 Magic Constant Summary Table

Table 1: Parameters of  $P(N, 1)$  for small odd  $N$ . The magic constant  $C = (19N+3)/2$  increases by 19 for each unit increase in  $m$  (equivalently, each increase of 2 in  $N$ ).

$N$ (odd), $m$	Vertices $p$	Edges $q$	Edge label range	Vertex label range	Magic constant $C$
$N = 3, m = 2$	$p = 2N = 6$	$q = 3N = 9$	$\{1, \dots, 9\}$	$\{10, \dots, 15\}$	$C = 30$
$N = 5, m = 3$	$p = 2N = 10$	$q = 3N = 15$	$\{1, \dots, 15\}$	$\{16, \dots, 25\}$	$C = 49$
$N = 7, m = 4$	$p = 2N = 14$	$q = 3N = 21$	$\{1, \dots, 21\}$	$\{22, \dots, 35\}$	$C = 68$
$N = 9, m = 5$	$p = 2N = 18$	$q = 3N = 27$	$\{1, \dots, 27\}$	$\{28, \dots, 45\}$	$C = 87$
$N = 11, m = 6$	$p = 2N = 22$	$q = 3N = 33$	$\{1, \dots, 33\}$	$\{34, \dots, 55\}$	$C = 106$
$N = 13, m = 7$	$p = 2N = 26$	$q = 3N = 39$	$\{1, \dots, 39\}$	$\{40, \dots, 65\}$	$C = 125$

## IX. DISCUSSION OF RESULTS

### 9.1 The Magic Constant Formula

The formula  $C = (19N+3)/2$  emerges naturally once the label ranges are fixed. Since edges occupy  $\{1, \dots, 3N\}$  and vertices occupy  $\{3N+1, \dots, 5N\}$ , the total edge-label sum is  $3N(3N+1)/2$  and the total vertex-label sum is  $N(8N+1)$ . Each edge appears in three magic-sum equations — one for each of its two endpoints, plus its own label — and each vertex label enters exactly three equations. Summing across all  $3N$  edges and solving gives the formula directly. The key structural observation is that the two ranges are complementary consecutive integer intervals, and this complementarity forces  $C$  to be an integer only when  $N$  is odd. There is no freedom in the value of  $C$ : it is uniquely determined by  $N$ .

### 9.2 Why Induction Adds Value

What sets this proof apart from Fukuchi (2001) is not the conclusion but the argument. A direct construction hands the reader a labeling formula and asks them to confirm it; the mechanism behind the formula remains opaque. The inductive argument

developed here reveals that mechanism explicitly: at each step, two new vertices are woven into the cycle structure, the existing labels shift upward uniformly, and the eight new edge slots absorb the smallest available labels. The magic constant rises by 19 because the shift equations require exactly  $2s_v + s_e = 19$  to hold at every retained edge, and the new vertex labels are then uniquely forced. This observation would be invisible in a direct construction but becomes the organising principle of the inductive proof.

A practical implication follows. Rather than looking up a formula valid for all  $N$ , one can build the labeling for any desired odd  $N$  step by step from the  $N=3$  base case, applying the vertex-insertion construction repeatedly. The process terminates after  $(N-3)/2$  steps and produces a verified labeling at each intermediate size as a byproduct.

### 9.3 Comparison with Prior Work

Table 2 situates this paper relative to Fukuchi (2001) across six criteria.

Table 2: Comparison of proof approaches

Criterion	Fukuchi (2001)	This paper
Proof method	Explicit construction	Mathematical induction
Magic constant	Formula given implicitly	Derived from algebra (Thm 4.3)
Base case	Not needed	$P(3,1)$ : 36 solutions by exhaustive search
Structural insight	None identified	Vertex-insertion mechanism
Extension scope	Case-specific	Template for $P(N,K)$ , $K>1$
Visual output	None	Verified diagrams for $N=3,5,7$

### 9.4 Limitations

The inductive proof of Section 6.2 guarantees existence: given a super magic labeling of  $P(2m-1,1)$ , one for  $P(2m+1,1)$  can be constructed. It does not, however, yield a single formula simultaneously valid for all odd  $N$ . For that, the reader is referred to Fukuchi (2001). Additionally, the random-restart search used for  $P(7,1)$  is probabilistic at the search stage, though the verification of the found labeling is entirely deterministic; the count of all super magic labelings of  $P(7,1)$  remains unknown.

## X. CONCLUSION AND FUTURE WORK

Three things have been established in this paper. The magic constant for  $P(N,1)$  with odd  $N$  is  $C = (19N+3)/2$ , derived here through a transparent algebraic argument that has not appeared elsewhere. The super magic property holds for all odd  $N \geq 3$ , proved by induction: the base graph  $P(3,1)$  carries 36 valid super magic labelings (found by exhaustive search in under one second), and the inductive step inserts two vertex pairs into the cycle structure while preserving all labeling requirements, increasing the magic constant by exactly 19 each time. Fully verified labelings for  $N = 3, 5$ , and  $7$  confirm the construction concretely, and rendered diagrams make the labeling structure accessible in a way that no purely algebraic presentation can match.

The structural contribution — identifying vertex insertion as the mechanism underlying the super magic property of  $P(N,1)$  — points to four specific research directions:

- i. Adapting the inductive framework to  $P(N, K)$  for  $K \geq 2$ , where the inner cycle spacing changes the insertion geometry.
- ii. Determining the super magic strength of  $P(N, 1)$  — the smallest achievable magic constant — and checking whether the inductive labeling achieves it.
- iii. Exploring analogous inductive constructions for super-edge-magic and antimagic labelings of  $P(N, K)$ .
- iv. Counting all distinct super magic labelings of  $P(N, 1)$  as a function of  $N$ , extending the result of 36 solutions for  $N = 3$ .

### ACKNOWLEDGMENTS

The authors are grateful to the anonymous referees for their careful reading and constructive feedback. All computations were performed using open-source software and are fully reproducible. The Python scripts underlying the verification and diagram generation are available from the

corresponding author (M. Ghaleppa, mallikarjun.g@utas.edu.au) upon request. No external funding was received for this study.

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**Citation of this Article:**

Mallikarjun Ghaleppa, Amit Kumar Yadav, Annadurai Manickam, Amabelle Oliva Enanoria, & Ramesh Palanisamy. (2026). Super Magic Labeling of Generalized Petersen Graphs  $P(N, 1)$ : An Induction-Based Proof with Computational Verification. *International Research Journal of Innovations in Engineering and Technology - IRJIET*, 10(6), 83-95. Article DOI <https://doi.org/10.47001/IRJIET/2026.106009>

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