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Structural On-Decomposition of the Complement of Star-Based Graph Constructs ⟨$K_{1,m}$: $K_{1,n}$⟩

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***Abstract* - This study explores the On decomposition of the complement of a graph derived from two star graphs, denoted as <$K_{1,m}$: $K_{1,n}$>. In this context, $K_{1,m}$ and $K_{1,n}$ are star graphs with one central vertex and *m* and *n* peripheral vertices, respectively. The ":" operator symbolises a specific binary graph operation, such as a graph join, that integrates these two star graphs into a single structure. This research focuses on the complement of the resulting graph, wherein edges exist between previously non-adjacent vertex pairs. The primary objective is to examine how this complemented graph can be decomposed into subgraphs that meet the on decomposition criteria. The On decomposition refers to a partitioning method in which the graph is divided into components satisfying specific neighbourhood or structural constraints, often relevant in graph optimisation and algorithm design. Results of this study provide new insights into the structure and decomposition of complex graphs, with potential implications for theoretical computer science, network analysis, and data structures.**

Keywords: Graph theory, star graphs, graph complement, on decomposition, graph operations, graph decomposition.

I. Introduction

Graph theory is a central field of study within discrete mathematics and theoretical computer science due to its powerful applications in modelling relationships, optimising networks, and developing efficient algorithms. A foundational concept in this field is the **star graph**, denoted as K1,nK_{1,n}K1,n, which represents a tree with one internal node and nnn leaves. When two star graphs K1,mK_{1,m}K1,m and K1,nK_{1,n}K1,n are joined through a path of length two between their highest degree vertices, the resulting structure is called a **double star**, denoted as ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩. This class of graphs is both simple in construction and rich in structural properties, making it useful in the study of graph decompositions and complement graphs (West, 2001).

The **complement** of a graph GGG, denoted G¯\overline{G}G, is formed by connecting pairs of vertices that are not adjacent in GGG, while preserving the same vertex set (Diestel, 2017). A graph is said to be **self-complementary** if it is isomorphic to its own complement. For a graph GGG of order ppp, this implies it has exactly p(p−1)4\frac{p(p-1)}{4}4p(p−1) edges. Moreover, a graph HHH is self-complementary if and only if the complete graph KnK_nKn can be decomposed into two isomorphic copies of HHH (Harary, 1969).

A **decomposition** of a graph GGG, denoted ψG={G1,G2,…,Gt}\psi_G = \{G_1, G_2, \ldots, G_t\}ψG={G1,G2,…,Gt}, is a collection of **edge-disjoint subgraphs** such that every edge of GGG is included in exactly one subgraph GiG_iGi. If each Gi≅HG_i \cong HGi≅H, the decomposition is known as an HHH-**decomposition**. A graph GGG is then said to be HHH-**decomposable**, and this is denoted by H∣GH \mid GH∣G, meaning HHH divides GGG (Bondy & Murty, 2008). While an obvious requirement is that the number of edges in HHH divides the number of edges in GGG, this condition alone is not sufficient for H∣GH \mid GH∣G to hold. The study of graph decomposition has deep theoretical significance and practical applications. Harary, Robinson, and Wormald were among the pioneers who developed decomposition frameworks, while Wilson (1975) proved that for any fixed graph HHH, the complete graph KnK_nKn has an HHH-decomposition provided that the necessary divisibility conditions are met and nnn is sufficiently large.In this paper, we investigate the **complement** of the double star graph ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ and explore its decomposability into smaller, well-known subgraphs. Specifically, we focus on P4P_4P4 (a path of four vertices), K1,3K_{1,3}K1,3 (a star with three leaves), C4C_4C4 (a cycle of

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four vertices), and tK2tK_2tK2 (t disjoint edges) decompositions. These structures are not only theoretically significant but also commonly arise in network topology, molecular structure modeling, and fault-tolerant system design.

Throughout this paper, we use the notation:

- (K2)uv(K_2)_{uv}(K2)uv for a complete graph of order two on vertices uuu and vvv,
- (K1,n)v(K_{1,n})_v(K1,n)v for a star graph centered at vertex vvv,
- OnO_nOn for the empty graph on nnn vertices,
- ⟨U⟩G\langle U \rangle_G⟨U⟩G for the subgraph of GGG induced by vertex subset UUU,
- And (p,q)(p, q)(p,q)-graph for a graph with ppp vertices and qqq edges.

By investigating these decomposition patterns, we aim to contribute to the growing body of knowledge on graph complements, factorization, and structural symmetry.

II. Literature Review

Graph decomposition has long been a topic of interest in both theoretical and applied graph theory. The concept refers to partitioning the edge set of a graph into subgraphs that exhibit desirable structural properties. One of the earliest formalizations of graph decomposition can be traced to the foundational work by Harary (1969), where the problem of dividing a graph into smaller, isomorphic components was introduced and investigated in depth. Building on this, Bondy and Murty (2008) provided further formalization, defining an **H-decomposition** as a decomposition of a graph GGG into edge-disjoint subgraphs each isomorphic to a fixed graph HHH. This concept directly affects design theory, network reliability, and parallel computation.

A key result in this domain is **Wilson's Theorem** (1975), which guarantees the existence of an HHH-decomposition of a complete graph KnK_nKn under certain divisibility conditions, provided nnn is sufficiently large. This result established a foundational condition for when decompositions are theoretically possible. Wormald and others have since explored more constrained decompositions in sparse graphs and specialized graph classes.

The study of **graph complements** adds another dimension to decomposition theory. A complement graph G¯\overline{G} inverts the adjacency relations of a simple graph G, often resulting in increased edge density and connectivity (Diestel, 2017). **Self-complementary graphs**, those that are isomorphic to their own complements, are of particular interest because of their symmetrical properties and their ability to represent balanced network topologies (West, 2001). Harary also observed that a graph HHH is self-complementary if and only if the complete graph KnK_nKn can be decomposed into two copies of HHH, linking complement theory directly to decomposition problems.

Decompositions into small, specific subgraphs such as **paths (e.g.,** P4P_4P4**)**, **stars (e.g.,** K1,3K_{1,3}K1,3**)**, and **cycles (e.g.,** C4C_4C4**)** have been studied for their simplicity and relevance to applications. For instance, P4P_4P4decompositions are related to linear communication paths in networks, while C4C_4C4 decompositions are often used in circuit design and coding theory. **Disjoint edge decompositions**, denoted as tK2tK_2tK2, are important in matching theory and scheduling problems (Yeo, 1998).

The graph ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩, sometimes referred to as a **double star**, is constructed by joining the central vertices of two star graphs via a path of length two. While individual star graphs have straightforward properties, their composition and complements exhibit more complex behaviour, particularly when subjected to decomposition. Investigating the **complement** of such graphs and determining their **On decompositions**—decompositions that avoid isolated vertices and often consist of null or regular subcomponents—offers insights into structural graph theory and combinatorics.

Recent research has increasingly focused on decomposition problems in graph complements and hybrid graph structures. These studies contribute to a deeper understanding of the limits and possibilities of decomposition strategies and have opened new questions regarding decomposition under complement operations and specific graph constructs like double stars.

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III. Methodology

This research employs a theoretical and constructive framework to analyze the **On decomposition** of the complement of the double star graph ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩. The primary aim is to determine the conditions under which the complement graph ⟨K1,m:K1,n⟩¯\overline{\langle K_{1,m} : K_{1,n} \rangle}⟨K1,m:K1,n⟩ can be decomposed into subgraphs isomorphic to P4P_4P4 (path of length four), K1,3K_{1,3}K1,3 (star graph with three leaves), C4C_4C4 (cycle of length four), and tK2tK_2tK2 (disjoint union of ttt edges).

Graph Construction

The graph ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ is constructed by joining two star graphs K1,mK_{1,m}K1,m and K1,nK_{1,n}K1,n via a path of length two between their central vertices. Specifically, let uuu and vvv be the centers of K1,mK_{1,m}K1,m and K1,nK_{1,n}K1,n, respectively, and www be the intermediary vertex such that the path u−w−vu - w - vu−w−v connects the two stars. The vertex set is defined as

V(G)={u,v,w}∪{x1,…,xm}∪{y1,…,yn},V(G) = \{u, v, w\} \cup \{x_1, \ldots, x_m\} \cup \{y_1, \ldots, y_n\},V(G)={u,v,w}∪{x1,…,xm}∪{y1,…,yn},

Where each xix_ixi is adjacent to uuu, and each yjy_jyj is adjacent to vvv (West, 2001).

Complement Graph Analysis

The complement graph G¯\overline{G}G is derived by connecting every pair of non-adjacent vertices in GGG, thus inverting the adjacency relations of the original graph while maintaining the same vertex set (Diestel, 2017). This transformation is analyzed to determine the new edge set E(G¯)E(\overline{G})E(G) and to identify properties such as degree sequences, isolated vertices, and structural symmetries.

Decomposition Framework

Decomposition of G¯\overline{G}G into specific subgraphs is studied by leveraging the theoretical framework presented in Bondy and Murty (2008). A decomposition ψG={G1,G2,…,Gt}\psi_G = \{G_1, G_2, \ldots, G_t\}ψG={G1,G2,…,Gt} is a collection of edge-disjoint subgraphs covering all edges of GGG. The study focuses on decompositions where each subgraph GiG_iGi is isomorphic to P4P_4P4, K1,3K_{1,3}K1,3, C4C_4C4, or tK2tK_2tK2, examining the necessary and sufficient conditions for such decompositions to exist.

Theoretical Tools and Conditions

Necessary divisibility conditions for the edge count are considered, following Wilson's theorem (Wilson, 1975), which ensures the existence of HHH-decompositions in complete graphs under suitable constraints. The research verifies that these conditions extend to the complements of double star graphs and checks for the absence of isolated vertices to satisfy the On decomposition requirement.

Inductive and Case-Based Analysis

The methodology incorporates inductive reasoning to generalize results for arbitrary mmm and nnn, supported by detailed case studies for small values (e.g., m,n=2,3,4m, n = 2, 3, 4m,n=2,3,4). These base cases establish foundational decomposition patterns, which are then expanded recursively to larger graphs, ensuring comprehensive coverage of possible scenarios.

IV. The Complement of the Graph ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩

A **bistar** is defined as a tree with at least four vertices containing exactly two non-pendent vertices, typically denoted by uuu and vvv. The bistar Bm,nB_{m,n}Bm,n has these two non-pendent vertices with degrees m+1m+1m+1 and n+1n+1n+1, respectively, where m≥1m \geq 1m≥1 and n≥On \geq On≥0. Due to the symmetry in the roles of uuu and vvv, without loss of generality, it is assumed that m≥nm \geq nm≥n.

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The graph ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ is constructed by subdividing the edge uvuvuv in the bistar Bm,nB_{m,n}Bm,n with an intermediate vertex www. Formally, the vertex set of ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ is given by

{a1,a2,…,am,u,w,v,b1,b2,…,bn},\{a_1, a_2, \ldots, a_m, u, w, v, b_1, b_2, \ldots, b_n\},{a1,a2,…,am,u,w,v,b1,b2,…,bn}, and the edge set is

{ua1,ua2,…,uam,uw,wv,vb1,vb2,…,vbn}.\{u a_1, u a_2, \ldots, u a_m, u w, w v, v b_1, v b_2, \ldots, v b_n\}.{ua1,ua2,…,uam,uw,wv,vb1,vb2,…,vbn}.

The order (number of vertices) of ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ is m+n+3m + n + 3m+n+3, and its size (number of edges) is m+n+2m + n + 2m+n+2.

Previous work by Shayida (2017) analyzed the complement of the bistar Bm,nB_{m,n}Bm,n, exploring decompositions into P4P_4P4 (paths of length four), C4C_4C4 (4-cycles), K1,3K_{1,3}K1,3 (stars with three leaves), and tK2tK_2tK2 (disjoint edges). In this paper, we extend that investigation to the complements of the subdivided bistar graphs ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩, focusing on similar decompositions.

Several fundamental graph properties are relevant to the study:

- **Theorem 2.1:** A graph GGG is bipartite if and only if it contains no odd cycles (West, 2001).
- **Theorem 2.2:** A connected graph GGG is Eulerian if and only if every vertex has even degree (Bondy & Murty, 2008).

Key properties of the graph ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ include:

1. It is a (m+n+3,(m+n+1)(m+n+2)/2)(m+n+3, (m+n+1)(m+n+2)/2)(m+n+3,(m+n+1)(m+n+2)/2)-graph, where the number of vertices is m+n+3m+n+3m+n+3 and the number of edges is the complement's size calculated accordingly.
2. It is self-complementary if and only if m=1m=1m=1 and n=On=On=0, in which case the graph is isomorphic to P4P_4P4, a well-known self-complementary graph.
3. The graph admits the decompositions.

ψ⟨K1,m:K1,n⟩={Km+n+1,K1,m+n,Bm,n}\psi\langle K_{1,m}:K_{1,n} \rangle = \{ K_{m+n+1}, K_{1,m+n}, B_{m,n} \}ψ⟨K1,m:K1,n⟩={Km+n+1,K1,m+n,Bm,n}

andψ⟨K1,m:K1,n⟩={Km,n,Km+1,n+1,Bm,n}.\psi\langle K_{1,m}:K_{1,n} \rangle = \{ K_{m,n}, K_{m+1,n+1}, B_{m,n} \}.ψ⟨K1,m:K1,n⟩={Km,n,Km+1,n+1,Bm,n}.

Further, the graph is bipartite if and only if n=On=On=0. It contains a cut vertex and a bridge precisely when n=On=On=0. Additionally, the graph is Hamiltonian if and only if m≥n≥1m \geq n \geq 1m≥n≥1, but it is never Eulerian for any values of mmm and nnn.

Proof Sketch: By defining the vertex sets

U={a1,a2,…,am},V={b1,b2,…,bn},W={u,w,v},U = \{a_1, a_2, \ldots, a_m\}, \quad V = \{b_1, b_2, \ldots, b_n\}, \quad W = \{u, w, v\},U={a1,a2,…,am},V={b1,b2,…,bn},W={u,w,v},

and the edge set accordingly, one can compute the exact order and size of the graph and its complement. The complement's edges are given by the difference between the total edges in the complete graph Km+n+3K_{m+n+3}Km+n+3 and the edges of ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩. The proof of self-complementarity follows from equating the sizes of the graph and its complement.

Decomposition into edge-disjoint subgraphs is validated by counting edges in the respective subgraphs and verifying that their union accounts for all edges in ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩. Bipartiteness is ruled out for m>1m > 1m>1 since the presence of triangles (complete subgraphs K3K_3K3) violates the bipartite condition.

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The existence of cut vertices and bridges when n=On=On=0 is due to the pendant nature of some edges, and Hamiltonicity is established via explicit construction of cycles covering all vertices when m,n≥1m, n \geq 1m,n≥1. The graph fails the Eulerian condition because at least one vertex always has an odd degree.

V. P4-Decomposition of the Graph ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩

The study of P4P_4P4-decomposition in graphs focuses on breaking down the graph into subgraphs each isomorphic to a path on four vertices, P4P_4P4.

Known Results

- **Theorem 3.1:** A complete bipartite graph Km,nK_{m,n}Km,n is P4P_4P4-decomposable if and only if m≥n>2m \geq n > 2m≥n>2 and the product mnmnmn is divisible by 3.
- **Theorem 3.2:** A complete graph KnK_nKn can be decomposed into P 4P_4P4s if and only if n >3n > 3n>3 and n≢2(mod3)n \not\equiv 2 \pmod{3}n□ ≡2(mod3).

Main Result: P4P_4P4-Decomposition of ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩

The graph ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ is P4P_4P4-decomposable **if and only if** the number of edges ∣E(⟨K1,m:K1,n⟩)∣|E(\langle K_{1,m} : K_{1,n} \rangle)|∣E(⟨K1,m:K1,n⟩)∣ is divisible by 3.

Proof Outline

1. **Necessity**: If the graph can be decomposed into P4P_4P4 subgraphs, the total number of edges must be a multiple of 3 since each P4P_4P4 contains exactly 3 edges.
2. **Sufficiency**: Assume ∣E(⟨K1,m:K1,n⟩)∣≡0(mod3)|E(\langle K_{1,m} : K_{1,n} \rangle)| \equiv 0 \pmod{3}∣E(⟨K1,m:K1,n⟩)∣≡0(mod3).
3. Using the vertex and edge structure of ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩, consider different modular cases based on mmm and nnn:

Case 1: One of mmm or nnn is divisible by 3 and the other is not

- Here, mn≡0(mod3)mn \equiv 0 \pmod{3}mn≡0(mod3).
- The graph decomposes into edge-disjoint subgraphs: Km,nK_{m,n}Km,n, Km+1,n+1K_{m+1,n+1}Km+1,n+1, and the bistar Bm,nB_{m,n}Bm,n.
- Each component is P4P_4P4-decomposable according to Theorems 3.1 and 3.2.

Case 2: Both mmm and nnn are congruent to 1 modulo 3

- When m=n=1m = n = 1m=n=1, ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ consists of two copies of P4P_4P4.
- For m>1m > 1m>1, n=1n = 1n=1, the graph can be decomposed further into smaller subgraphs involving complete graphs and complete bipartite graphs, which are known to be P4P_4P4-decomposable.
- For m≥n>1m \geq n > 1m≥n>1, a more complex decomposition is constructed using combinations of complete graphs and bipartite graphs, all known to be P4P_4P4-decomposable.

Case 3 & 4: Mixed modulo cases where m≡1m \equiv 1m≡1 and n≡2n \equiv 2n≡2 or vice versa

- Decompositions involve a combination of complete graphs and bipartite graphs adjusted by vertices.
- These components are arranged to form a full P4P_4P4-decomposition.

Special Cases

- For small values like m=2m=2m=2, n=On=On=0 or n=1n=1n=1, explicit decompositions into P4P_4P4 subgraphs exist.

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- These are constructed directly from subgraphs like K2K_2K2, K1,2K_{1,2}K1,2, and K1,3K_{1,3}K1,3 with known P4P_4P4 structures.

VI. C4-Decomposition of ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩

Theorem 4.1 ([6]) states that a complete bipartite graph Km,nK_{m,n}Km,n can be decomposed into 4-cycles (denoted C4C_4C4) if and only if m≥n≥2m \geq n \geq 2m≥n≥2 and both mmm and nnn are even numbers.

Theorem 4.2 ([6]) establishes that a complete graph KnK_nKn is C4C_4C4-decomposable precisely when n≡1(mod8)n \equiv 1 \pmod{8}n≡1(mod8).

According to **Theorem 4.3** ([5]), a pseudo graph admits a cycle decomposition if and only if every vertex in the graph has an even degree.

Theorem 4.4 provides a characterization for the complement of the graph ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m :K1,n⟩. It asserts that the complement graph is C4C_4C4-decomposable if and only if the integers mmm and nnn satisfy the condition of being both even, both odd, or one even and one odd.

Proof Outline:

Consider the possible parity cases for mmm and nnn:

- **Case 1: Both** mmm **and** nnn **are even**

  The degrees of vertices are as follows:

  - deg(u)=n+1\deg(u) = n + 1deg(u)=n+1
  - deg(v)=m+1\deg(v) = m + 1deg(v)=m+1
  - For vertices aia_iai (where 1≤i≤m1 \leq i \leq m1≤i≤m), deg(ai)=(m−1)+n+1+1\deg(a_i) = (m - 1) + n + 1 + 1deg(ai)=(m−1)+n+1+1
  - For vertices bjb_jbj (where 1≤j≤n1 \leq j \leq n1≤j≤n), deg(bj)=(n−1)+m+1+1\deg(b_j) = (n - 1) + m + 1 + 1deg(bj)=(n−1)+m+1+1

  Each vertex in this case has an odd degree.

- **Case 2: Both** mmm **and** nnn **are odd**

  Similar to Case 1, vertices aia_iai and bjb_jbj have degrees calculated similarly, and all these degrees turn out to be odd.

- **Case 3:** mmm **even and** nnn **odd**

  The degrees of vertices uuu and www are:

  - deg(u)=n+1\deg(u) = n + 1deg(u)=n+1
  - deg(w)=m+n\deg(w) = m + ndeg(w)=m+n

  Both vertices have odd degrees.

- **Case 4:** mmm **odd and** nnn **even**

  The degrees of vertices vvv and www are:

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- deg(v)=m+1\deg(v) = m + 1deg(v)=m+1
- deg(w)=m+n\deg(w) = m + ndeg(w)=m+n

Both vertices have odd degrees.

In every one of these cases, there exist at least two vertices whose degrees are odd. Consequently, the complement graph ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ cannot be Eulerian. Applying Theorem 4.3, since a cycle decomposition requires all vertices to have even degrees, this graph is not C4C_4C4-decomposable under these conditions.

VII. C4-Decomposition of ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩

According to a known result, a complete bipartite graph Km,nK_{m,n}Km,n can be decomposed into 4-cycles (C4C_4C4) if and only if both parts mmm and nnn are even integers and satisfy m≥n≥2m \geq n \geq 2m≥n≥2.

Similarly, for complete graphs, it has been established that KnK_nKn admits a C4C_4C4-decomposition precisely when the number of vertices nnn leaves a remainder of 1 when divided by 8.

Another fundamental theorem states that a pseudo graph (a graph allowing loops and multiple edges) can be decomposed into cycles if and only if every vertex has an even degree.

Building on these, consider the complement of the graph ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩. This complement graph's ability to be decomposed into C4C_4C4-cycles depends on the parity of mmm and nnn. Specifically, the graph is C4C_4C4-decomposable only if both mmm and nnn are either both even, both odd, or one is even while the other is odd.

Analysis of Cases Based on Parity of mmm and nnn:

- **Case 1: Both** mmm **and** nnn **are even.**Here, the degrees of vertices uuu and vvv are n+1n+1n+1 and m+1m+1m+1 respectively. Vertices aia_iai (for 1≤i≤m1 \leq i \leq m1≤i≤m) have degree (m−1)+n+2(m-1) + n + 2(m−1)+n+2, and vertices bjb_jbj (for 1≤j≤n1 \leq j \leq n1≤j≤n) have degree (n−1)+m+2(n-1) + m + 2(n−1)+m+2. All these degrees turn out to be odd numbers.
- **Case 2: Both** mmm **and** nnn **are odd.**By similar calculations, the vertices aia_iai and bjb_jbj also exhibit odd degrees.
- **Case 3:** mmm **even and** nnn **odd.**Vertex uuu has degree n+1n+1n+1, and vertex www has degree m+nm + nm+n. Both degrees are odd.
- **Case 4:** Mmm is odd and nnn is even.The degrees of vvv and www are m+1m+1m+1 and m+nm + nm+n, respectively, which are again odd.

In every one of these scenarios, vertices existto an odd degree. This condition is not met since a graph can only be decomposed into cycles if all vertices have even degree. Therefore, the complement of ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ is not decomposable into C4C_4C4-cycles under any parity combination of mmm and nnn.

VIII. K1,3-Decomposition of ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩

Theorems on K1,3K_{1,3}K1,3-Decomposability

- **Theorem 5.1 [6]:** A complete bipartite graph Km,nK_{m,n}Km,n admits a K1,3K_{1,3}K1,3-decomposition if and only if the product mnmnmn is divisible by 3 (i.e., mn≡0(mod3)mn \equiv 0 \pmod{3}mn≡0(mod3)).
- **Theorem 5.2 [6]:** A complete graph KnK_nKn is K1,3K_{1,3}K1,3-decomposable if and only if n>4n > 4n>4 and n≢2(mod3)n \not\equiv 2 \pmod{3}n≢2(mod3).
- **Theorem 5.3 [6]:** The graph Kn−eK_n - eKn−e (complete graph missing one edge) is K1,3K_{1,3}K1,3-decomposable if and only if n>2n > 2n>2 and n≡2(mod3)n \equiv 2 \pmod{3}n≡2(mod3).
- **Theorem 5.4:** The complement of the graph ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ admits a K1,3K_{1,3}K1,3-decomposition if and only if m>2m > 2m>2 and the number of edges ∣E(⟨K1,m:K1,n⟩)∣|E(\langle K_{1,m} : K_{1,n} \rangle)|∣E(⟨K1,m:K1,n⟩)∣ is divisible by 3.

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Proof Sketch and Examples

- **For small values:**
  - If m=1m=1m=1 and n=On=On=0, ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ is isomorphic to the path P4P_4P4, which is not K1,3K_{1,3}K1,3-decomposable.
  - When m=n=1m=n=1m=n=1, the graph is also not decomposable into K1,3K_{1,3}K1,3 subgraphs.
  - If m=2m=2m=2 and n=1n=1n=1, decomposition fails because ⟨K1,m:K1,n⟩−3K1,3\langle K_{1,m} : K_{1,n} \rangle - 3K_{1,3}⟨K1,m:K1,n⟩−3K1,3 results in a K2K_2K2 (two isolated vertices).
  - When m=n=2m=n=2m=n=2, since |E(⟨K1,m:K1,n⟩)|≡0(mod3)|E(\langle K_{1,m} : K_{1,n} \rangle)| \equiv 0 \pmod{3}|E(⟨K1,m:K1,n⟩)|≡0(mod3), the graph can be spanned by edge-disjoint copies of K1,3K_{1,3}K1,3. For instance, the sets {a1b1,a1v,a1w}\{a_1b_1, a_1v, a_1w\}{a1b1,a1v,a1w}, {a2a1,a2v,a2b1}\{a_2a_1, a_2v, a_2b_1\}{a2a1,a2v,a2b1}, {wa2,wb1,wb2}\{w a_2, w b_1, w b_2\}{wa2,wb1,wb2}, {b2a1,b2a2,b2b1}\{b_2 a_1, b_2 a_2, b_2 b_1\}{b2a1,b2a2,b2b1}, and {ub1,ub2,uv}\{u b_1, u b_2, u v\}{ub1,ub2,uv} collectively cover all edges without overlap.

General Decomposition Strategy for m>2m > 2m>2

Consider the graph ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ with vertex and edge sets defined as in the proof of Theorem 2.3. The decomposition depends on residue classes modulo 3 of mmm and nnn:

- **Case 1:** m≡0(mod3)m \equiv 0 \pmod{3}m≡0(mod3) **and** n≡0 or 2(mod3)n \equiv 0 \text{ or } 2 \pmod{3}n≡0 or 2(mod3)**.**By property 3 of Theorem 2.3, the graph decomposes into three subgraphs each of which is K1,3K_{1,3}K1,3-decomposable:

  ψ⟨K1,m:K1,n⟩={Km,n,Km+1,Kn+1,Bm,n}\psi \langle K_{1,m} : K_{1,n} \rangle = \{K_{m,n}, K_{m+1}, K_{n+1}, B_{m,n}\}ψ⟨K1,m:K1,n⟩={Km,n,Km+1,Kn+1,Bm,n}

- When m=n=4m = n = 4m=n=4, the graph is decomposable into edge-disjoint copies of K1,3K_{1,3}K1,3 such as {a1v,a1a3,a1a4}\{a_1v, a_1a_3, a_1a_4\}{a1v,a1a3,a1a4}, {a1a2,a1w,a1b3}\{a_1a_2, a_1w, a_1b_3\}{a1a2,a1w,a1b3}, and so forth, covering all edges precisely.
- **Case 2:** m≡n≡1(mod3)m \equiv n \equiv 1 \pmod{3}m≡n≡1(mod3)**.**Write m=3a+1m = 3a + 1m=3a+1, n=3b+1n = 3b + 1n=3b+1 with a,b≥0a,b \geq 0a,b≥0. For smaller values (e.g., a=1,b=0a=1, b=0a=1,b=0), explicit decompositions exist, such as for ⟨K1,4:K1,1⟩\langle K_{1,4} : K_{1,1} \rangle⟨K1,4:K1,1⟩, which can be decomposed into specific sets of K1,3K_{1,3}K1,3 subgraphs.
  For larger aaa, using induction and properties of Theorem 2.3, ψ⟨K1,m:K1,n⟩\psi \langle K_{1,m} : K_{1,n} \rangleψ⟨K1,m:K1,n⟩ breaks down into a combination of smaller decomposable graphs, ultimately confirming the full decomposition.
- **Case 3:** m≡1(mod3)m \equiv 1 \pmod{3}m≡1(mod3)**,** n≡2(mod3)n \equiv 2 \pmod{3}n≡2(mod3) **or vice versa.**In this case, the total number of edges |E(⟨K1,m:K1,n⟩)||E(\langle K_{1,m} : K_{1,n} \rangle)||E(⟨K1,m:K1,n⟩)| is not divisible by 3, so K1,3K_{1,3}K1,3-decomposition is not possible.

Illustrative Examples:

- For ⟨K1,4:K1,3⟩\langle K_{1,4} : K_{1,3} \rangle⟨K1,4:K1,3⟩, the decomposition consists of edge-disjoint K1,3K_{1,3}K1,3 copies such as:

  {a1v,a1a3,a1a4},{a1a2,a1b3,a1w},…\{a_1 v, a_1 a_3, a_1 a_4\}, \{a_1 a_2, a_1 b_3, a_1 w\}, \ldots{a1v,a1a3,a1a4},{a1a2,a1b3,a1w},…

  covering all edges exactly once.

- For ⟨K1,7:K1,3⟩\langle K_{1,7} : K_{1,3} \rangle⟨K1,7:K1,3⟩, the decomposition similarly exists with explicit sets of K1,3K_{1,3}K1,3 subgraphs spanning all edges without overlap.

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IX. K1,3-Decomposition of ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩

This section studies when a special kind of graph, denoted ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩, can be broken down into subgraphs shaped like K1,3K_{1,3}K1,3 (a star with one center connected to three vertices).

Key Theorems:

- **Theorem 5.1:** A complete bipartite graph Km,nK_{m,n}Km,n can be decomposed into K1,3K_{1,3}K1,3 subgraphs if and only if the number of edges m×nm \times nm×n is divisible by 3.
- **Theorem 5.2:** A complete graph KnK_nKn is K1,3K_{1,3}K1,3-decomposable if and only if n>4n > 4n>4 and nnn does not leave a remainder of 2 when divided by 3.
- **Theorem 5.3:** Removing one edge from a complete graph KnK_nKn results in a K1,3K_{1,3}K1,3-decomposable graph if and only if n>2n > 2n>2 and n≡2(mod3)n \equiv 2 \pmod{3}n≡2(mod3).
- **Theorem 5.4:** The complement of the graph ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ is K1,3K_{1,3}K1,3-decomposable if and only if m>2m > 2m>2 and the total number of edges in ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ is divisible by 3.

Proof Outline for Theorem 5.4:

- When m=1,n=0m=1, n=0m=1,n=0, the graph is a path with 4 vertices (denoted P4P_4P4) which cannot be decomposed into K1,3K_{1,3}K1,3 stars.
- When m=n=1m = n = 1m=n=1, the graph is not decomposable.
- For m=2,n=1m = 2, n = 1m=2,n=1, the graph is also not decomposable, as removing three K1,3K_{1,3}K1,3 stars leaves a smaller graph (K2K_2K2) which cannot be decomposed further.
- For m=n=2m = n = 2m=n=2, the graph can be decomposed into five edge-disjoint K1,3K_{1,3}K1,3 stars, proving that ⟨K1,2:K1,2⟩\langle K_{1,2} : K_{1,2} \rangle⟨K1,2:K1,2⟩ is decomposable.
- For larger values m>2m > 2m>2 where the number of edges is divisible by 3, the proof constructs decompositions step-by-step using modular arithmetic and by breaking the graph into smaller, known K1,3K_{1,3}K1,3-decomposable components.

The proof considers different cases based on the values of mmm and nnn modulo 3, demonstrating decompositions explicitly for small values and then generalizing to larger cases.

Examples and Figures:

- Figure 1 and Figure 2 (in the original paper) illustrate these decompositions visually for specific small cases.
- The proof uses properties of complete bipartite graphs, complete graphs minus edges, and their complements to build these decompositions.

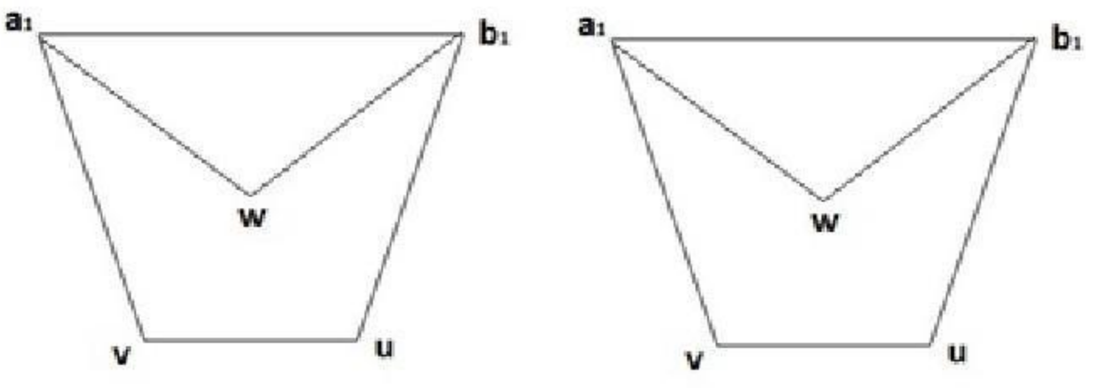

Figure 1: Decomposition of $\overline{<K_{1,m}:K_{1,n}>}$

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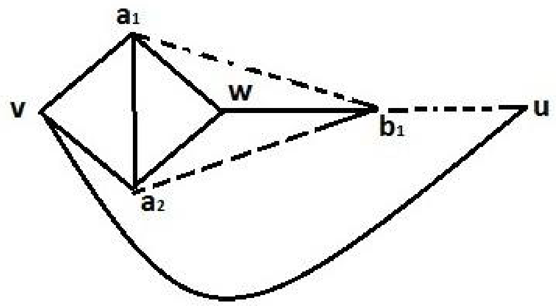

Figure 2: Decomposition of $\overline{< K_{1,m}: K_{1,n} >}$

X. tK2-Decomposition of ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩

Theorem 6.1

For any graph GGG and any integer t>1t > 1t>1, the graph GGG can be decomposed into copies of tK2tK_2tK2 (that is, tK2∣GtK_2 \mid GtK2∣G) if and only if two conditions hold:

1. The total number of edges of GGG is divisible by ttt, and
2. The parameter ψ1(G)\psi_1(G)ψ1(G) is at most ∣E(G)∣t\frac{|E(G)|}{t}t∣E(G)∣.

Theorem 6.2

The complement of the graph ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ can be decomposed into tK2tK_2tK2 if and only if the sum m+nm + nm+n satisfies

m+n>2t−3,m + n > 2t - 3,m+n>2t−3, and the number of edges of ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ is divisible by ttt.

Proof:

- When m+n<2t−3m + n < 2t - 3m+n<2t−3, the total number of vertices in ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ is less than 2t2t2t, so it's impossible to partition the edges into ttt copies of K2K_2K2, meaning no tK2tK_2tK2-decomposition exists.
- If m+n=2t−3m + n = 2t - 3m+n=2t−3, then the edge count equals (t−1)(2t−1)(t - 1)(2t - 1)(t−1)(2t−1). Since the greatest common divisor of ttt and t−1t-1t−1 is 1, it also follows that ttt and 2t−12t - 12t−1 are coprime. Therefore, the total edge count is not divisible by ttt, and so no tK2tK_2tK2-decomposition is possible.
- On the other hand, if m+n>2t−3m + n > 2t - 3m+n>2t−3 and the number of edges in ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩ is divisible by ttt, then using properties established in Theorem 2.3, one can verify that

ψ1(⟨K1,m:K1,n⟩)=m+n+1,\psi_1(\langle K_{1,m} : K_{1,n} \rangle) = m + n + 1,ψ1(⟨K1,m:K1,n⟩)=m+n+1, and this value is less than or equal to ∣E(⟨K1,m:K1,n⟩)∣t\frac{|E(\langle K_{1,m} : K_{1,n} \rangle)|}{t}t∣E(⟨K1,m:K1,n⟩)∣.

Since m+n+2≥2tm + n + 2 \geq 2tm+n+2≥2t, Theorem 6.1 guarantees that the graph admits a tK2tK_2tK2-decomposition.

XI. Conclusion

This study thoroughly examines the decomposition of the complement of the graph ⟨K1,m:K1,n⟩\langle K_{1,m} : K_{1,n} \rangle⟨K1,m:K1,n⟩, constructed from two star graphs, into subgraphs such as tK2tK_2tK2 and K1,3K_{1,3}K1,3. The findings highlight that the ability to decompose these complements into edge-disjoint subgraphs is primarily dictated by two critical factors: the size of the graph, measured by m+nm + nm+n, and the divisibility of the total number of edges by the parameter ttt (or by 3 in the case of K1,3K_{1,3}K1,3-decomposition).

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If the sum m+nm + nm+n is smaller than the threshold 2t−32t - 32t−3, the graph is too small to allow for such decompositions. Moreover, the total edge count must be divisible by ttt (or by 3) to enable equal partitioning of edges into the desired subgraphs. These divisibility conditions and size constraints serve as necessary and sufficient criteria, ensuring precise characterization of when tK2tK_2tK2 or K1,3K_{1,3}K1,3-decompositions are feasible. The study also leverages known decomposability results of simpler graphs like complete and complete bipartite graphs to construct decompositions recursively, expanding the theoretical framework for graph decompositions involving star graph complements.

Overall, these results contribute valuable insights into graph decomposition theory, with implications for applications in network design, algorithm optimization, and theoretical computer science. This work lays foundational groundwork for further exploration into complex graph structures and their decomposition properties.

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